DERIVATIVE PRICING

A TECHNICAL GUIDE, PART I

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List of Abbreviations

The following abbreviations and symbols are used in this guide:

AAF	Annuity adjustment factor
ac.APM	The option pricing tool described in these notes
ac.SRM	The interest rate derivative pricing tool
APM	Asset price model
ATM	At-the-money, the situation when the price of the underlying security equals the strike price
ATMF	At-the-money forward option, the situation when the price of the underlying security equals the option strike price on the forward market
bps	Basis point (same as pip, equal 1/100th of 1%)
BS	Black-Scholes
BSM	Black-Scholes-Merton
BV	Book Value
CCY	Currency
CDS	Credit default swap
DC	Deal contingent
DTCC	Depository Trust And Clearing Corporation
ES	Expected shortfall
FMV	Fair market value
FV	Forward value
FXIP	FX Information Portal Bloomberg function
G/L	Gain and loss
M&A	Merger and acquisition
MTM	Marked-to-market
OCI	Other comprehensive income
OECD Guidelines	"BEPS Actions 8 – 10, Financial Transactions", a draft published in July – September 2018 for the purposes of public discussion
OV	Bloomberg option valuation tool
P&L	Profit and loss
Pip	Percentage in point (same as bps, equal 1/100th of 1%)
PP	Purchase Price
PPE	Purchase price equation
SRM	Short rate model of interest rates
VaR	Value-at-risk

Section 1 Introduction

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1.1 Applications of derivative pricing in transfer pricing

Derivative pricing based on theoretical financial models is not a direct benchmarking approach which is generally preferred in transfer pricing analysis. It is applied in cases when it is not feasible to identify comparable transactions with publicly available market prices. The approach based on the fair market valuation (**FMV**) of derivative instruments is typically classified as "Other Methods" under the hierarchy of transfer pricing methods.

After the FMV approach is selected as a transfer pricing valuation model (under the "Other Methods" category), a further review is required to identify a correct FMV model. The choice of the model depends on the derivative type and review of risks of the counterparties in the derivative contracts.

All models discussed in this guide were applied in a specific transfer pricing project. In this section, the context, in which the models were applied, is summarized in the format of a collection of stylized transfer pricing examples.

Example 1: **Valuation of convertible bonds**.

Example 2: **Valuation of downstream loan guarantee**.

Example 3. **Valuation of upstream guarantee**.

Example 4: **Valuation of commodity call option**.

Example 5: **Valuation of residual value risk contracts**.

Example 6: **Share purchase commitment**.

Example 7: **Capacity guarantees**.

1.2 Terminology

The following terminology is used I the guide.

- Black-Scholes model – derivative pricing model for the underlying asset which price follows geometric Brownian motion.
- Black model – derivative pricing model for the bond underlying asset
- CDS model – option valuation model in which movement in the asset value is related to the asset default state
- Default hazard rate – marginal probability of default conditional on survival event
- Implied default hazard rate – the default hazard rate derived from the market bond data
- Implied premium schedule
- Arrow-Debreu prices – prices of basis states which do not intersect and span the universe of all possible states. Arrow-Debreu prices are used as a basis to derive the price of an arbitrary derivative instrument.
- Risk-neural probabilities – forward value (FV) of the Arrow-Debreu prices. FV adjustment is applied to ensure that risk-neutral probabilities sum up to one.
- Annuity adjustment factor (**AAF**) – the factor applied to convert fixed price into the equivalent sequence of periodic payments
- Interest rate parity – refers to equation (2.10) used to derive the FX forward price.
- Option intrinsic value – the difference between the spot and strike price: $V = S - K$
- Option time value – the difference between the option value and the option intrinsic value. The option price is presented as a sum of option intrinsic value and time value.
- FX swap – a combination of FX spot and forward contracts
- Cross-currency swap – interest rate swap of interest rates denominated in different currencies.
- Worst Loss – the highest potential loss in the transaction risk exposure. The metrics is applied as a risk measure in the risk assessment analysis.

1.3 Arbitrage vs insurance pricing approaches

The arbitrage pricing and insurance are two conceptually different approaches to price guarantee/insurance contracts. The arbitrage approach views the contract as a derivative instrument and prices it based on market prices. Insurance approach views the contract as an insurance agreement and prices it based on historical data and respective expected losses and return to risk components.

Despite conceptual differences of the two approaches, the pricing equations under two approaches can look very similar. The difference typically stems from the interpretation of the equation parameters and not the form of the equation. Under the insurance approach parameters are derived based on statistical analysis applied to historical data while under the arbitrage approach the parameters are derived based on market price data and referred typically as 'implied' parameter.[1] For example, under the insurance approach the risk of default on a specific debt instrument would be estimated using historical default data while under

[1] Implied parameter refers to parameter implied by market data as opposed to estimated from historical data.

the arbitrage approach the risk of default would be estimated based on the instrument market risk premium. The default rate would be derived from the market premium and interpreted as default rate ‘implied’ by market pricing.

In most cases, in transfer pricing analysis the arbitrage pricing approach is preferred over the insurance approach. The highest ranking is assigned in transfer pricing analysis to the pricing method which is based on a search for comparable transactions traded in the market and using market data to price the tested transaction. In finance arbitrage pricing refers to a more generic approach which may be based directly on identifying direct comparable instruments but may also apply pricing methods based on complex trading strategies of the identified instruments.

It is recommended when possible to apply both the insurance and arbitrage pricing approaches and compare the results of the analysis. Insurance approach is based exclusively on historical data while the arbitrage approach, which is based on market prices, takes into account forward-looking expectations of the agents trading in the market. Therefore, comparison of the results under the two approaches effectively implies comparison of the effect that historical and forward-looking information have on the results. The purpose of the comparison is to identify what different information is contained in the forward-looking expectations compared to the historical data. It’s natural to expect reasonably similar results under the two approaches in stable markets and different results in volatile markets or markets which go through significant transformations.

This guide provides several examples which compare the results of valuation analysis performed under the two approaches.

Section 2 Models

The stock, commodity, and FX prices (denoted as S_t) are typically modelled using geometric Brownian motion, which is described by the following stochastic differential equation

(2.1) $$S_{t+dt} = S_t \times e^{\mu dt + \sigma \sqrt{dt} \varepsilon_t}$$

where $\varepsilon_t \sim N(0,1)$. Parameters μ and σ are interpreted as drift and volatility parameters. The equation can be equivalently represented as follows.

(2.2) $$\Delta \ln S_t = \mu dt + \sigma \sqrt{dt} \varepsilon_t$$

The following notation is used in the price equations below:

1. T is the derivative contract maturity term
2. t is a date between zero and maturity term T
3. S_t is the spot price in period t; S is the spot price in period $t = 0$
4. K is the derivative contract strike price
5. σ is the spot price volatility (measured as standard deviation of $\Delta \ln S_t$ process)
6. r is the risk-free rate
7. d is the dividend rate (in FX or commodity price derivative contracts parameter d is interpreted respectively as commodity lease rate (or negative of commodity storage cost) and risk-free of return in foreign country)

2.1 Stocks

The stock price is assumed to pay regular dividends at annual rate d ($d = \frac{D}{S}$, where D is fixed annual dividend payment).

2.1.1 Futures

Definition: futures contract

(2.3) $$F_T = S \times e^{(r-d) \times T}$$

2.1.2 Forwards

Definition: forward contract

In forwards, the strike price K can be set different from the futures price F_t. The value of the forward contract in period t is

(2.4) $$V_t = e^{-r(T-t)} \times \left(S_t \times e^{(r-d)(T-t)} - K\right) = S_t \times e^{-d(T-t)} - Ke^{-r(T-t)}$$

2.1.3 Deal contingent forwards

Definition: A deal contingent (**DC**) forward is a specialised forward FX contract. The hedging buyer is only obliged to fulfil the contract if a planned major transaction, such as an acquisition, occurs. The DC forward requires no payment upfront, locks in a forward rate, and disappears with zero fee if the M&A deal fails.

The DC forward contracts are issued with banks. A bank must have the following characteristics to issue a DC contract.[2]

1. Risk appetite and balance sheet exposure to the FX risk. There's no traded market in deal-contingent hedges, so the bank can't lay off this risk.
2. Comprehensive M&A knowledge to analyze the purchase agreement and conditions precedent. The bank needs to have specialist M&A expertise and hedging capabilities to offer competitive hedge pricing even when they have no role in the underlying M&A.
3. The ability to competitively and consistently price hedges, execute them effectively and quantify and manage the risk associated with them (which requires awareness of likely drivers of FX movements and interest rates). The bank must have an established transaction history in order to achieve a competitive price.
4. The ability to coordinate multiple units across the bank (as well as their legal division) in order to optimize the speed of the DC forward execution (timing is an important factor in the execution of DC forward contracts).

As illustrated in the example discussed in Appendix **Error! Reference source not found.**, the ATM put option is the optimal strategy from the Worst Loss risk measure metrics. Therefore, if a risk manager is considering a hedging strategy that is not based on deal contingent instruments, the put option price is the minimum price that provides a 100% risk exposure hedging.

The objective of the DC forward contracts traded in the OTC markets is to reduce the hedging costs by (i) reducing the upside gain exposure; and (ii) by moving some of the risk to the bank that sells the contract. Because the ATM put option is effectively the upper bound for the DC forward, the premium on the DC forward is estimated in percentage of the put option premium. The actual percentage negotiated by the bank and the contract buyer depends on the risk appetite of the bank and the buyer, the specific facts and circumstances of the deal, as well as other factors. The price of the DC forward is expressed by the following equation.

(2.5) $$F_T^{DC} = F_T + \alpha \times P_T^{put}$$

where P_T^{put} is estimated as a percentage of the forward notional amount and parameter α is typically set within $[40\%, 70\%]$ range.

[2] https://www.nomuraconnects.com/focused-thinking-posts/deal-contingent-hedging-a-flexible-way-to-mitigate-risk/.

2.1.4 Call and Put options

Definition: call

Definition: put

Definition: **warrant** is a security that entitles the holder to buy the underlying stock of the issuing company at a fixed price called exercise price until the expiry date.

Call option

(2.6) $$V^{call} = N(d_1) \times Se^{-dT} - N(d_2) \times Ke^{-rT}$$

Put option

(2.7) $$V^{put} = -N(-d_1) \times Se^{-dT} + N(-d_2) \times Ke^{-rT}$$

where parameters d_1 and d_2 are estimated as

(2.8) $$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}} \times \left[\ln\frac{S}{K} + \left(r - d + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

An alternative presentation of the call and put option equations is presented in (4.11) - (4.13). The alternative representation substitutes spot rates with forward rates (using equation (2.3)). Note that option prices in the alternative presentation do not depend on dividend rate d parameter.

To ensure consistency of equations (2.6) - (2.8) with alternative representation of the equations (4.11) - (4.13), the dividend rate in the equations (2.6) - (2.8) is estimated as the implied dividend based on the equation (2.3):

(2.9) $$d = r - \frac{\ln\frac{F_T}{S}}{T}$$

If implied dividend rate set based on equation (2.9), then the futures are priced consistently with call and put options.

2.2 Commodities

2.3 Currency FX

Suppose that the FX spot and forward rates are quoted relative to the local currency (we assume USD as the local currency). Specifically, S is the current spot price and F_t is the forward price of one unit of foreign currency in local currency units (e.g. USD/CAD is the number of units of USD currency paid for one CAD currency unit).[3]

2.3.1 Forwards

The price of a forward is described by the following equation:

(2.10) $$F_t = S \times e^{(r - r_f) \times t}$$

where r is the local (US) risk-free rate, r_f is the risk-free rate in the foreign country, and t is the maturity term of the forward contract. The equation is similar to the forward price equation for the stock forward if the dividend rate is replaced by the r_f parameter. The foreign currency can be viewed as an asset which is similar to stock but has r_f as an equivalent of dividend rate.

Equation (2.10) is a well-known **interest rate parity** relationship which can be interpreted as follows from the arbitrage-free pricing perspective.

Arbitrage interpretation of equation (2.10). (i) Convert 1 USD into $\frac{1}{S}$ CAD, (ii) invest $\frac{1}{S}$ CAD to generate risk free return $\frac{1}{S} \times e^{r_f \times t}$ CAD; (iii) convert it back to $\frac{F_t}{S} \times e^{r_f \times t}$ USD; (iv) verify that the generated return equals to the return on 1 USD: $\frac{F_t}{S} \times e^{r_f \times t} = e^{r \times t}$.

For a FX forward contact with F_0 fixed forward rate, the fair market value at a given period t is equal to

(2.11) $$P = S \times e^{r_f \times t} - F_0 \times e^{-r \times t} = (F_t - F_0) \times e^{-r \times t}$$

2.3.2 Call and put options

The equations for the FX call and put options are derived similarly to the equations for the stock call and put options by replacing dividend rate in equations (2.6) - (2.8) with the r_f parameter.[4]

Call option

(2.12) $$V^{call} = N(d_1) \times Se^{-r_f T} - N(d_2) \times Ke^{-rT}$$

Put option

(2.13) $$V^{put} = -N(-d_1) \times Se^{-r_f T} + N(-d_2) \times Ke^{-rT}$$

where parameters d_1 and d_2 are estimated as

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[3] $S^{USD/CAD} = 0.7$ implies that 0.75 USD dollars are paid for 1 CAD dollar.

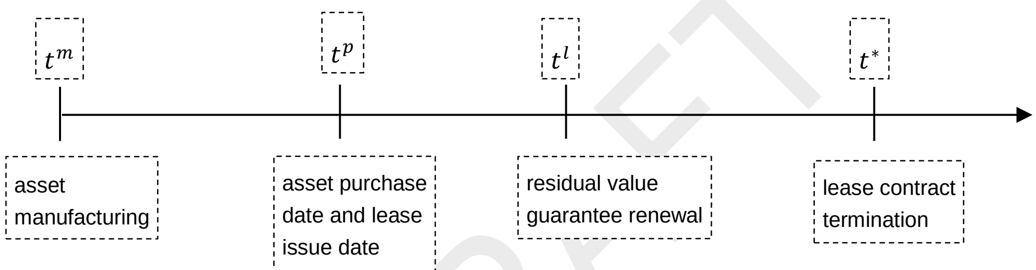
[4] For additional details see https://en.wikipedia.org/wiki/Foreign_exchange_option.

$$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}} \times \left[\ln\frac{S}{K} + \left(r - r_f + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases} \quad (2.14)$$

2.4 CDS

2.5 Leases

Lease timeline



Typically, all lease contracts in the portfolio are renewed at the same date (date t^l is the same for each lease contract).

2.5.1 Terminology

The following terminology is used in the leases fee calculations:[5]

1. Capitalized Cost – the cost of the vehicle after subtracting any down payment or trade-in allowance. The capitalized cost is denoted as X^C;
2. Residual Value – the amount the vehicle is worth at the end of the lease. The residual value is denoted as X^R;
3. Depreciation – the amount the vehicle has lost in value during the lease. Depreciation is calculated as $X^D = X^C - X^R$;
4. Lease Term – the number of months until the lease contract is terminated. The lease term is denoted as T;
5. Money Factor (or Lease Factor) – the finance charge, usually expressed as a fraction. The money factor is denoted as f;
6. US railcar retrofit schedule – The U.S. Department of Transportation has released a final rule unveiling a new enhanced tank car standard and risk-based retrofitting schedule for older tank cars

[5] http://www.realcartips.com/leasing/0434-how-to-calculate-lease-payments.shtml

carrying crude oil and ethanol. While certain tank cars in crude oil service must be retrofitted as soon as 2017, the initial retrofit requirements for tank cars carrying ethanol don't begin until 2023[6].

2.5.2 Model

The lease model consists of the following components:

(i) Calculation of lease monthly payments;

(ii) Estimation of the lease asset residual value;

(iii) Lease value stochastic model

Each component is reviewed in detail below.

2.5.2.1 Lease monthly payments

The monthly lease payment is calculated in the lease model as follows. The lessor acquires the car at price X^C and predicts that the cost at the end of the lease will be equal to X^R. The monthly lease amount consists of the following components: (i) compensation for the car depreciation; (ii) lease interest payment; and (iii) lease taxes. The three components of the total lease payment ($F = F^D + F^I + F^T$) are calculated as follows:

1. Depreciation compensation. The monthly payment is calculated directly as $F^D = \frac{X^C - X^R}{T}$;
2. Interest. The interest is calculated as $F^I = f \times (X^C + X^R) = (2f) \times \frac{X^C + X^R}{2}$. Note that $2f$ is a monthly lease fee factor and $2f \times 1{,}200 = f \times 2{,}400$ is interpreted as the annual interest rate (calculates as % of 100 nominal amount). The value $\frac{X^C + X^R}{2}$ is interpreted as the average car value during the lease term. The interest expense is calculated as the percentage of the average car value.
3. Taxes. The taxes are calculated as a fixed percentage of the before-tax lease payment: $F^T = t \times (F^D + F^I)$.

2.5.2.2 Lease residual value

The following residual value specification is used in these notes:

(2.15)
$$V_t = V_0 \times e^{-\int_0^t \lambda(s)ds + \sigma W_t} = V_0 \times e^{-\Lambda_t + \sigma W_t}$$

where W_t is a diffusion process. Under the model specification, the residual value decreases exponentially over time and is described by geometric Brownian motion. Deviation of the actual depreciation from the expected depreciation natural logarithm is described by a diffusion process:

(2.16)
$$v_t = v_0 - \Lambda_t + \sigma W_t$$

where $v_t = \ln V_t$. We refer to function Λ_t as the **expected depreciation term structure**. The function Λ_t may have a general non-linear specification so that it can produce an arbitrary depreciation term structure.[7]

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[6] http://ethanolproducer.com/articles/12189/dot-rule-includes-new-tank-car-standards-retrofit-schedule

[7] The function $\Lambda_t = -\ln(1 - \lambda t)$ produces a standard book-value expected linear depreciation term structure: $V_t = V_0 \times (1 - \lambda t)$.

There are other alternative models of the leased asset residual value. Below we describe briefly some alternative model specifications.

[summary of literature]

2.5.3 Modelling residual risk

There are two alternative approaches to residual value risk, which are discussed in these notes.

1. Residual value risk, which results from the destruction of the underlying asset value. The risk is modelled using a binomial model with the absorbing state (which corresponds to the destruction of the underlying asset). The hazard rate of the asset destruction is described by function $\gamma(s)$.
2. Residual value risk, which results from the accelerated depreciation of the underlying market asset (compared to the expected depreciation). The accelerated depreciation can be due to technological changes, asset obsolescence risk, change in the regulations, and other. The risk is modelled using the geometric Brownian motion specification of the residual value (described in Section 2.1.2.2).

The third alternative approach is to use a combination of the two above residual value models.

Under the first approach, the model of the lease residual value guarantee is similar the credit default swap (**CDS**) pricing model. Under the first approach model of the lease residual value guarantee is similar the Black-Scholes (**BS**) put option model. A more detailed discussion of the alternative lease residual value guarantee models is presented in the sections below.

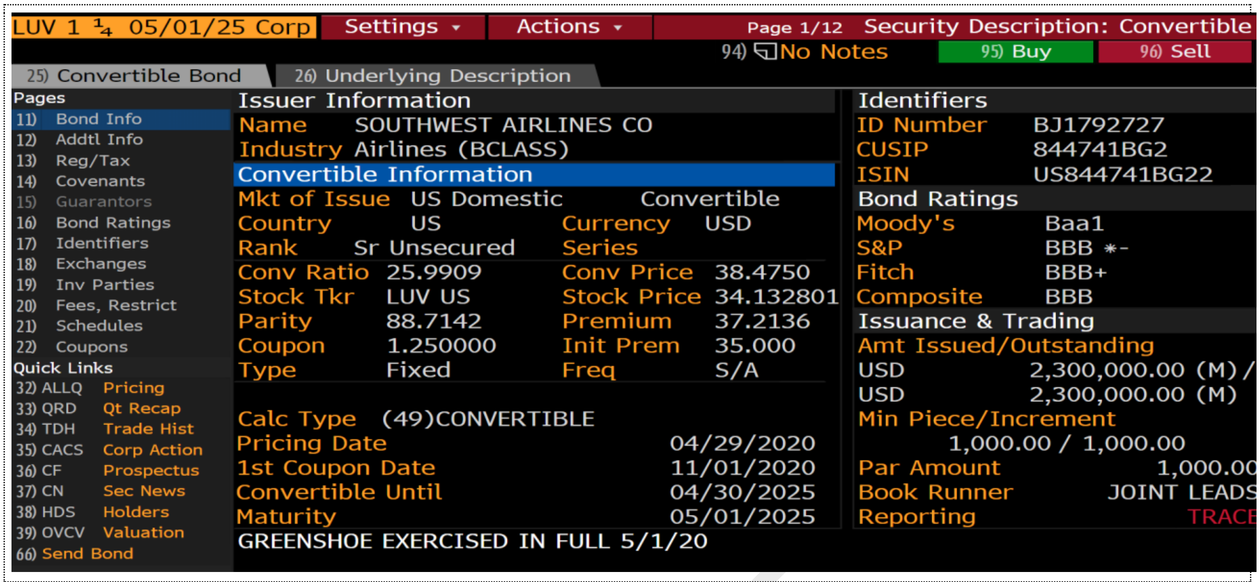
2.6 Bonds

2.6.1 Convertible bonds

Definition: A convertible bond is a fixed-income corporate debt security that yields interest payments but can be converted into a predetermined number of common stock or equity shares. The conversion from the bond to stock can be done at certain times during the bond's life and is usually at the discretion of the bondholder.

The terms of a standard convertible bond as presented on Bloomberg are shown in the exhibit below (the print screen was produced on 22 June 2020).

Exhibit 2.1 Terms of a convertible bond transaction

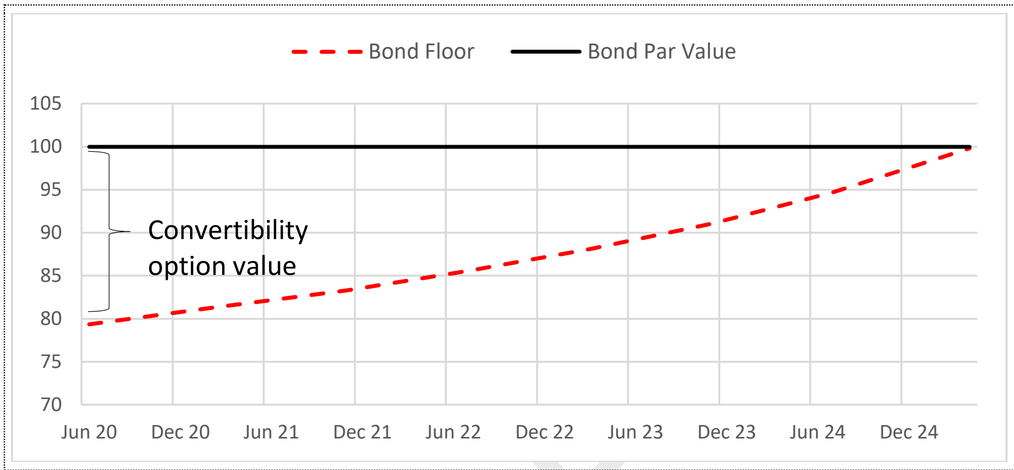


The terms of the convertible bond are interpreted as follows.

1. **Par Amount (A = 1,000)**: the notional bond amount as of issue date. For convenience, the amount is presented in fixed 1,000 units. The terms of convertible bond are described respectively in terms of conversion of the bond 1,000 par value into shares.
2. **Conversion Price (P)**: share price applied to convert bond par value into shares.
3. **Conversion Ratio (R = A/P)**: number of shares into which bond par value is converted.
4. **Last conversion date**: the latest date when the bond can be converted into stock (typically set one day prior to the maturity date).
5. **Stock Price (S)**: current stock price.
6. **Parity (= S x P / A)**: Immediate value of the convertible if converted, typically obtained as current stock price multiplied by the conversion ratio expressed for a base of 100. May also be known as Exchange Property.
7. **Premium**: defined as current convertible price minus the parity. The premium can also be expressed as a percentage of parity. In the example above, the market price of the bond is 121.728. The premium equals 33.099 (= 121.728 – 88.714) or 37.214% (= 33.099 / 88.714).
8. **Initial Premium**: premium at the bond issuance date.
9. **Bond floor**: Value of the fixed income element of a convertible i.e. not considering the ability to convert into equities.
10. **Call features**: The ability of the issuer (on some bonds) to call a bond early for redemption. This should not be mistaken for a call option. A Softcall would refer to a call feature where the issuer can only call under certain circumstances, typically based on the underlying stock price performance (e.g. current stock price is above 130% of the conversion price for 20 days out of 30 days). A Hardcall feature would not need any specific conditions beyond a date: that case the issuer would be able to recall a portion or the totally of the issuance at the Call price (typically par) after a specific date.

The diagram below shows the break-down of the bond total value into the bullet bond value (bond floor) and the value of the convertibility option. The diagram shows that in the presence of material value of the convertibility option, the coupon rate can be set significantly lower than the discount rate applicable to the bon so that the bullet bond is priced significantly below the par value. The reduced value of the coupon payments is compensated to the lender by the option to convert the bond into the shares if the price of the shares increases in the future.[8]

Exhibit 2.2 Break down of bond value into (i) bond floor and (ii) convertibility option value



Note that if the convertibility option is not deeply out-of-the-money, the yield adjustment for bond convertibility can be quite material. To perform the adjustment, the following steps need to be performed: (i) the value of the convertibility option is estimated; (ii) the option value is subtracted from the bond price; and (iii) the yield rate is estimated based on the adjusted bond price.

Convertible bonds are typically issued for companies with high risk and high potential growth. Low coupon payments allow the company which issues a convertible bond mitigate the risk of default at early growth stage when the earnings are low. The benefit to the lender is generated by the company growth potential.

The details of convertibility option valuation are discussed in Section 4.4.3 and in Appendix **Error! Reference source not found.**. Some key considerations related to the convertible bond valuation are summarized below.

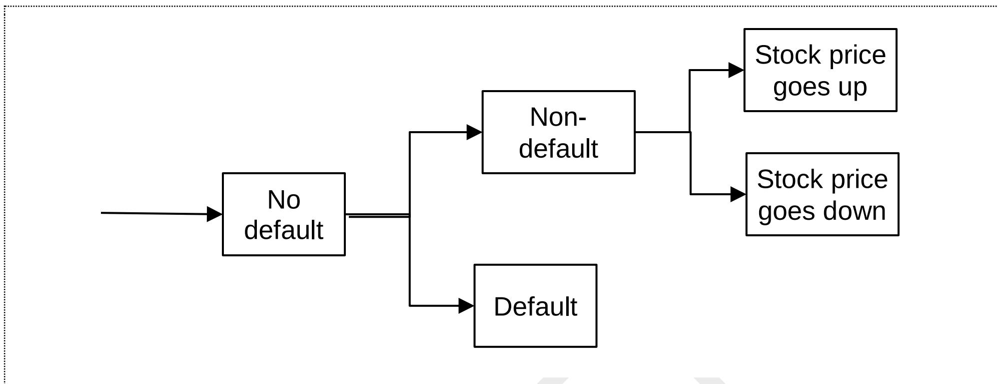
1. If convertible bond is deep **out-of-the-mone**y (stock price is significantly lower than the strike price), then the option value can be assumed approximately equal to zero.
2. All terms of the convertible bond should be considered since the terms such as soft call may have a significant impact on the value of convertibility option.
3. The convertible bonds are typically issued to reduce the fixed income payouts of the bond instrument and compensate it with the potential benefit of the stock price upward movement. As a result, a convertible bond is a **hybrid** instrument which has both the features of bond and equity.
4. The adjustment for convertibility option can be very **significant** and the convertibility option can effectively determine the bond value.
5. The valuation of the convertibility option can be **sensitive** to the underlying assumptions and change materially with the change in the assumptions.

[8] In the example, the bond is traded at par. Note that the price is determined by the market and can be above / below par starting from the bond issue date.

6. Due to properties described in items 4 and 5, it is generally recommended to **exclude** convertible bonds from the sample (unless the bond is deep out-of-the-money and convertibility option can be ignored).
7. After performing the adjustment for bond convertibility option, the yield on the bond will generally be higher than the yield on a comparable bond issued by the same company. This is due to hybrid feature and higher risk of a convertible bond.

The convertible bond can be modelled using the following 3-state and 3-asset model:

Exhibit 2.3 Tree modelling of convertible bond process



The three assets are represented by a risk-free bond, non-convertible bond price and stock price.

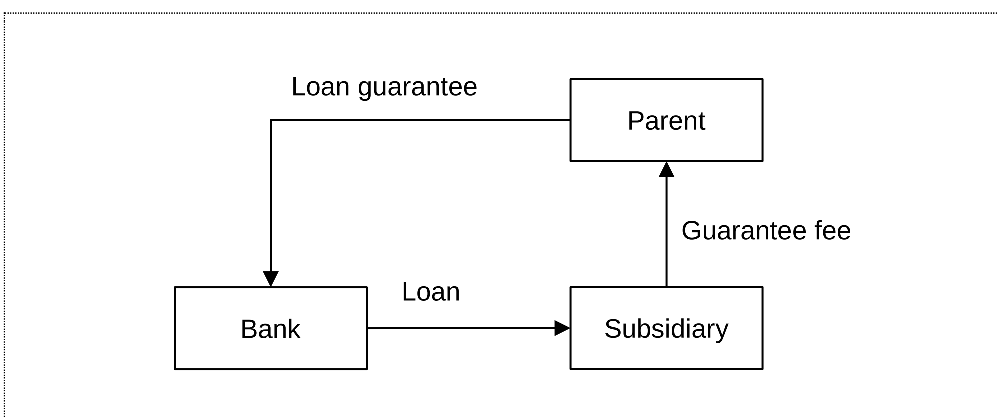
2.7 Loan guarantee models

A loan guarantee model is an extension of a CDS model which is described by a trinomial tree. The additional states are included to (i) either model counterparty (guarantor) risk in a downstream loan guarantee arrangement or (ii) model uncertain recovery rate on defaulted loan in an upstream loan guarantee arrangement.

2.7.1 Downstream loan guarantee

The downstream loan guarantees are discussed in detail in the ‘Financial Guarantees’ guide. This guide presents the model from the perspective of derivative pricing and risk-neutral valuation. Schematically, the downstream guarantee model is described by the following diagram.

Exhibit 2.4 Structure of a downstream loan guarantee transaction

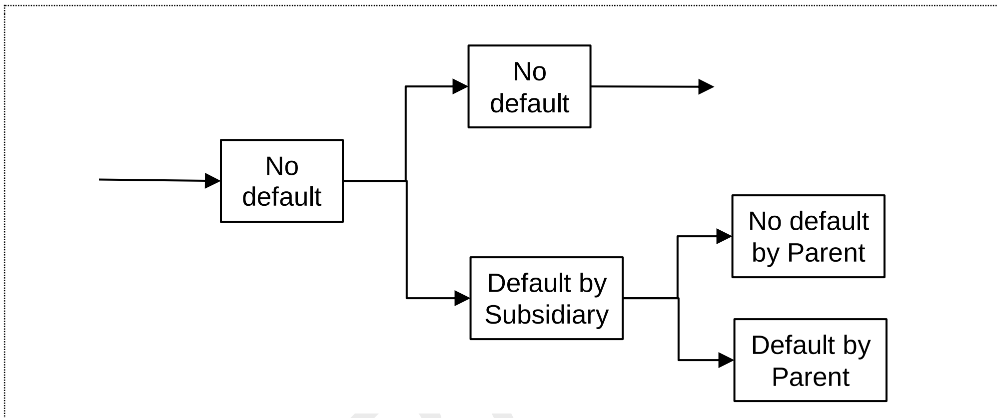


In a downstream loan guarantee transaction, the guarantee is provided on by the Parent entity to the Bank for the Loan made to the Subsidiary entity. The problem is described by a three-state three-asset model, where the three states are (i) no default, (ii) default by the Subsidiary, and (iii) default by both the Parent and the Subsidiary; and three assets are (i) risk-free loan, (ii) loan issued by the Subsidiary on a stand-alone basis, and (iii) loan issued by the Parent group.

In a downstream loan guarantee model, the guarantee is provided by the Parent group for the purpose of the Subsidiary credit enhancement and respective reduction in the borrowing costs. Effectively, the guaranteed loan can be viewed as the loan made to the Parent group. The model is similar to a CDS model but also takes into consideration the possibility of the Parent group default (counterparty risk).

From the theoretical perspective, the downstream loan guarantee can be modelled as a sequence of binary models described by the following diagram.

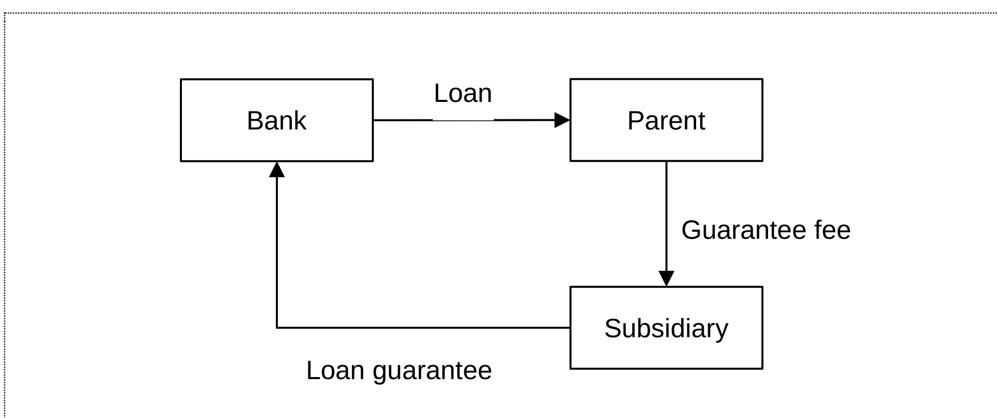
Exhibit 2.5 Tree modelling of a downstream loan guarantee



2.7.2 Upstream loan guarantee

The difference of an upstream from the downstream loan guarantee is that the loan made to the Parent is guaranteed by Subsidiary entity. The purpose of the explicit loan guarantees is to simplify to the lender access to the Subsidiary assets in the even of default by the Parent group. Schematically, the downstream guarantee model is described by the following diagram.

Exhibit 2.6 Structure of an upstream loan guarantee transaction

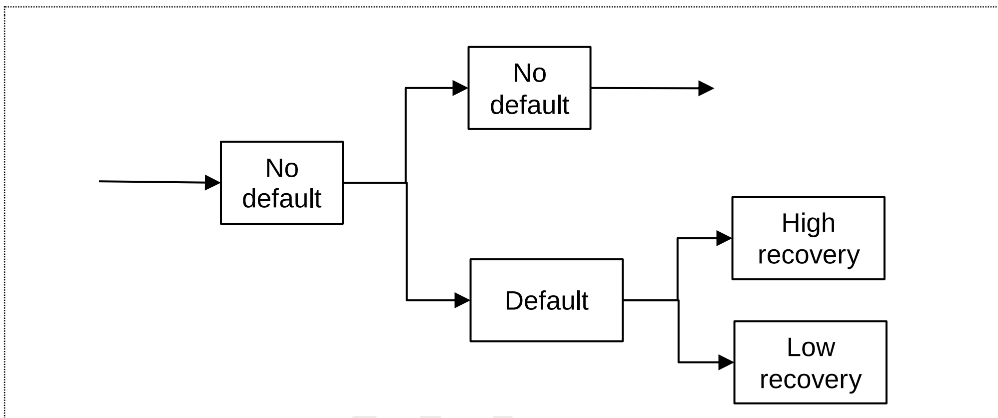


The problem is described by a three-state three-asset model, where the three states are (i) no default, (ii) default by the Parent, high recovery on defaulted loan, and (iii) default by both the Parent and low recovery on defaulted loan; and three assets are (i) risk-free loan, (ii) loan issued by the Subsidiary on a stand-alone basis, and (iii) loan issued by the Parent group.

The model assumes that the recovery on a non-guaranteed loan is uncertain and can be either high or low while in the case of guaranteed loan the recovery is assumed to be always high. The approach models a higher certainty in the access of the lender to the assets of the Subsidiary and, therefore, lower uncertainty in the recovery on the defaulted loan.

From the theoretical perspective, the upstream loan guarantee can be modelled as a sequence of binary models described by the following diagram.

Exhibit 2.7 Tree modelling of an upstream loan guarantee



In practice, upstream guaranteed typically involve multiple subsidiaries and effectively present cross-guarantees between the Parent group subsidiaries some of which are borrowers, and some are guarantors in the loan transaction. The guarantee fee is allocated respectively from the net borrowers to net guarantors in the group. The model with two counterparties discussed in this guide is applied as an important building block of the cross guarantees model with multiple counterparties. The model of cross guarantees is discussed in more detail in the 'Cooperative Games' guide.

2.8 Share purchase commitment

[Canopy]

Section 3 Risk-Neutral Probabilities

A risk-neutral measure (also called an equilibrium measure, or equivalent martingale measure) is a probability measure such that each financial instrument price is exactly equal to the discounted expectation of the instrument cash flows under this measure. Such a measure exists if and only if the market is arbitrage-free. The risk-neutral measure is unique whenever the market is **complete**.

Risk-neutral probabilities are closely related to Arrow-Debreu prices, which are the prices of basic financial instruments which pay $1 dollar in a specific state and zero in any other state of the underlying stochastic model of the financial market. The Arrow-Debreu price equals to the respective discounted risk-neutral probability of the state.

Conceptually, the risk-neutral probabilities are derived as follows. The prices of the financial instruments traded in the market are used to derive Arrow-Debreu prices. The transformation from the prices of traded instruments into the Arrow-Debreu prices exists and unique whenever the markets are arbitrage=free and complete. The risk-neutral probabilities are estimated from the Arrow-Debreu prices directly as discussed above. The risk-neutral probabilities are applied then to derive the price of an arbitrary derivative instrument.[9]

3.1 General model specifications

In this section, the risk-neutral probabilities are derived for general model specifications. In the following sections the formulas are applied to derive prices of specific financial instruments.

3.1.1 Multi-state model

In this section, the risk-neutral distribution equations for the case of n –state and n –traded securities are derived. The general case is presented first to introduce notation and to describe the equations in general matrix form. Special cases of the general model are described in the following sections and applied to derivatives for specific financial instruments.

The model is presented for two periods but is extended directly to arbitrary number of periods. Suppose that $S = (S_1, \dots, S_n)$ is the vector of asset prices and $P = P(S)$ is an $n \times n$ matrix with the payoff structure of n assets, where P_{ij} is the payoff of security $i = 1, \dots, n$ in state $j = 1, \dots, n$.

Suppose that $\mathcal{A} = (\mathcal{A}^1, \dots, \mathcal{A}^n)$ represent a portfolio of n **Arrow-Debreu securities** with the payoff matrix equal to identity matrix (denoted as I). The **replication portfolio** Λ for the Arrow-Debreu securities is estimated from the following linear equation

(3.1) $$\Lambda \times P = I$$

or

[9] The instrument is referred to as a derivative since its cash flows and respectively price are derived from other instruments which are traded and priced by the market.

$$\sum_i \Lambda_{ki} \times P_{ij} = I_{kj}$$

where Λ_{ki} is the number of shares of asset i acquired to replicate Arrow-Debreu security k. Assuming that the rank of matrix P equals n, the replication portfolio is estimated from the following equation:

(3.2) $$\Lambda = P^{-1}$$

The **Arrow-Debreu prices** are described by equation

(3.3) $$A = P^{-1} \times S$$

and **risk-neutral probabilities** are described by equation

(3.4) $$Q = A \times (1 + R) = P^{-1} \times S \times (1 + R)$$

Throughout the guide, unless specified explicitly, we assume the following interpretation of indices:

1. Symbol i refers to an underlying asset index
2. Symbol j refers to a state index
3. Symbol k refers to an Arrow-Debreu security index
4. Symbol n refers to the total number of underlying assets / states / Arrow-Debreu securities.

3.1.2 Binary model

Binary model is used as a basic building block for the discrete and continuous models. The binary model is described by two periods $t = 0,1$ and two price states S^u, S^d.[10] The initial price state is denoted as S, initial risk-free bond price is denoted as B, and risk-free rate of return is denoted as R. The stock dividend paid during the period is denoted as D. The two instruments are denoted respectively as $\mathcal{S}$ and $\mathcal{B}$. In matrix form, the payoff function can be presented as follows

(3.5) $$\mathrm{P} = \begin{pmatrix} S^u & S^d \\ B(1+R) & B(1+R) \end{pmatrix}$$

Suppose that $\mathcal{A}^u = (1,0)$ and $\mathcal{A}^d = (0,1)$ denote Arrow-Debreu assets. Suppose also that $\mathcal{A}^u$ is replicated as $\mathcal{A}^u = \alpha^u \mathcal{S} + \beta^u \mathcal{B}$. In the matrix form the equation is presented as follows:

$$\begin{cases} \alpha^u(S^u + DS) + \beta^u B(1+R) & = & 1 \\ \alpha^u(S^d + DS) + \beta^u B(1+R) & = & 0 \end{cases}$$

The solution of the system of equations for the replication portfolio is described as

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[10] For convenience, we use notation (u, d) for states $j = 1,2$, where state u corresponds to upward movement and state d corresponds to downward movement.

(3.6)
$$\begin{cases} \alpha^u = \dfrac{1}{(S^u - S^d)} \\ \beta^u = \dfrac{1}{B(1+R)} \times \dfrac{-(S^d + DS)}{(S^u - S^d)} \end{cases}$$

Similarly,

(3.7)
$$\begin{cases} \alpha^d = \dfrac{-1}{(S^u - S^d)} \\ \beta^d = \dfrac{1}{B(1+R)} \times \dfrac{S^u + DS}{(S^u - S^d)} \end{cases}$$

In matrix form the replication portfolio is presented as follows:

(3.8)
$$\Lambda = \begin{pmatrix} \alpha^u & \beta^u \\ \alpha^d & \beta^d \end{pmatrix} = \frac{1}{(S^u - S^d)} \times \begin{pmatrix} 1 & \dfrac{-(S^d + DS)}{B(1+R)} \\ -1 & \dfrac{S^u + DS}{B(1+R)} \end{pmatrix}$$

The Arrow-Debreu prices, denoted as A^u and A^d, are calculated as

(3.9)
$$\begin{cases} A^u = \alpha^u S + \beta^u B = \dfrac{S(1+R-D) - S^d}{(S^u - S^d)(1+R)} \\ A^d = \alpha^d S + \beta^d B = \dfrac{-S(1+R-D) + S^u}{(S^u - S^d)(1+R)} \end{cases}$$

or in the vector notation

(3.10)
$$\mathrm{A} = \frac{1}{(S^u - S^d)(1+R)} \times \begin{pmatrix} S(1+R-D) - S^d \\ -S(1+R-D) + S^u \end{pmatrix}$$

The Arrow-Debreu prices have the following property: $A^u + A^d = \frac{1}{1+R}$.[11] Therefore, the normalized Arrow-Debreu prices can be interpreted as risk-neutral probabilities (denoted respectively as q^u and q^d):

(3.11)
$$\begin{cases} q^u = A^u(1+R) = \dfrac{S(1+R-D) - S^d}{(S^u - S^d)} \\ q^d = A^d(1+R) = \dfrac{-S(1+R-D) + S^u}{(S^u - S^d)} \end{cases}$$

or in vector notation

[11] The property follows directly from the fact that $q^u B(1+R) + q^d B(1+R) = B$ (risk-neutral price of the risk-free bond equals the actual price of the risk-free bond).

(3.12)
$$Q = \frac{1}{(S^u - S^d)} \times \begin{pmatrix} S(1+R-D) - S^d \\ -S(1+R-D) + S^u \end{pmatrix}$$

The probabilities q^u and q^d are called risk-neural due to the following property: the expected risk-neutral price of the underlying asset equals to

(3.13)
$$F = E^Q[S] = q^u S^u + q^d S^d = S(1+R-D) = S(1+R) - D^F$$

where $D^F = SD$ is fixed dividend paid in period t.

The price F is also interpreted as the forward price of the asset.

The process risk-neutral standard deviation is calculated using the following equation:[12]

(3.14)
$$\sigma^{2,Q}[S] = q^u S^{2,u} + q^d S^{2,d} - (q^u S^u + q^d S^d) = (S^u - S^d) q^u q^d$$

3.1.3 Sequence of binary models

In this section, we derive the equations for n –state (with the focus on 3 –state) model which can be reduced to a sequence of binary models. The reduction of n –state model to sequence of binary models can be performed when the payoffs of n assets have the following property.

Sequential payoff structure property. We say that the assets have a sequential payoff structure, if the payoff matrix of n assets has the following property.

(3.15) For each $i = 1, \dots, n$, asset A^i is an instrument with constant payoff in states $j = i, \dots, n$

Asset A_1 is the risk-free asset which has a constant payoff in each state $j = 1, \dots, n$.

For consistency with the notation in Section 3.1.2, the states $j = 1,2,3$ are also denoted as $j = u, du, dd$, where u is up state, du is down state followed by up state and dd is down state followed by down state. The model has the following applications:

1. **Convertible bond**. Three assets are represented by risk-free bond, risky bond, and stock, where the risky bond and stock are issued by the company. Three states are represented by default state ($j = 1$), non-default state with stock moving up ($j = 2$), and non-default state with stock moving down ($j = 3$). The payoff of the risky bond is assumed to be constant in non-default state.
2. **Downstream loan guarantee**. Three assets are represented by risk-free bond, bond issued by the parent entity, and bond issued by subsidiary entity. Three states are represented by non-default state ($j = 1$), default state by subsidiary only ($j = 2$), and default state by both parent and subsidiary ($j = 3$). The payoff of the bond issued by the parent is assumed to be constant if parent did not default. The payoff on the bond issued by subsidiary may be different depending on whether only subsidiary entity or both the parent and subsidiary entities have defaulted.

[12] The equation also applies to actual standard deviation by replacing the risk-neutral probabilities q^u and q^d with actual probabilities.

3. **Upstream loan guarantee**. Three assets are represented by risk-free bond, non-guaranteed bond issued by the group, and guaranteed bond issued by the group. Three states are represented by non-default state ($j = 1$), default and high recovery on defaulted loan ($j = 2$), and default and low recovery on defaulted loan ($j = 3$). The payoff of the guaranteed loan is assumed to be constant in both low and high recovery states of defaulted loan. The payoff on non-guaranteed loan is different in the low and high recovery states of defaulted loan.

The reduction to a sequence of binary model is illustrated below for the 3-security and 3-state example with the securities denoted as B (risk-free security), S^1 (security with constant payoffs in states $j = 2,3$), and S^2 (security with a different payoff in each state). The argument tis easily extended to the n −state model. The application of the model to convertible bond and intercompany financial guarantee valuation is discussed in Sections 4.4.3 and 4.3.3.

The replication of Arrow-Debreu securities is performed in two steps:

1. In step 1, securities with the following payoffs structure are replicated using assets $i = 1,2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

   where the first security is Arrow-Debreu security$\mathcal{A}^1$ and the second security is denoted as $\mathcal{A}^{2,3}$. Note that replicated payoff in state $j = 3$ will be exactly the same as payoff in state $j = 3$ due to property (3.15). Therefore, it is sufficient to replicate the payoff structure in states $j = 1,2$ using the two assets $i = 1,2$.

2. In step 2, securities with the following payoff structure are replicated using asset $\mathcal{A}^{2,3}$ and asset $i = 3$:

$$\begin{pmatrix} x & y \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   where x and y are some arbitrary numbers. The securities are denoted respectively as $\tilde{\mathcal{A}}^1$ and $\tilde{\mathcal{A}}^2$. The numbers are converted to zero by adding respective shares of Arrow-Debreu security $\mathcal{A}^1$ constructed in step 1. Security $\mathcal{A}^{2,3}$ is viewed as a risk-free security at the stage two of the binary model estimation process.

In matrix form, the estimation process can be described as follows. Suppose that

$$\Lambda^1 = \begin{pmatrix} \alpha^{1,u} & \beta^{1,u} \\ \alpha^{1,d} & \beta^{1,d} \end{pmatrix} \text{ and } \Lambda^2 = \begin{pmatrix} \alpha^{2,u} & \beta^{2,u} \\ \alpha^{2,d} & \beta^{2,d} \end{pmatrix}$$

are the 2x2 matrix estimated at stages one and two. In a 3x3 matrix representation the step 1 and 2 equations are represented as follows

$$\begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^{2,3} \\ S^2 \end{pmatrix} = \begin{pmatrix} \alpha^{1,u} & \beta^{1,u} & 0 \\ \alpha^{1,d} & \beta^{1,d} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} B \\ S^1 \\ S^2 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -x & \alpha^{2,u} & \beta^{2,u} \\ -y & \alpha^{2,d} & \beta^{2,d} \end{pmatrix} \times \begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^{2,3} \\ S^2 \end{pmatrix}$$

Combining the two equations, the equation for the Arrow-Debreu Securities can be represented as follows:

$$\begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -x & \alpha^{2,u} & \beta^{2,u} \\ -y & \alpha^{2,d} & \beta^{2,d} \end{pmatrix} \times \begin{pmatrix} \alpha^{1,u} & \beta^{1,u} & 0 \\ \alpha^{1,d} & \beta^{1,d} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} B \\ S^1 \\ S^2 \end{pmatrix}$$

where parameters x and y are selected so that the first element of the $\mathcal{A}^2$ and $\mathcal{A}^3$ vectors is equal to zero. The equation can be simplified as follows:

$$\begin{pmatrix} \mathcal{A}^1 \\ \mathcal{A}^2 \\ \mathcal{A}^3 \end{pmatrix} = \begin{pmatrix} \alpha^{1,u} & \beta^{1,u} & 0 \\ -x\alpha^{1,u} + \alpha^{2,u}\alpha^{1,d} & -x\beta^{1,u} + \alpha^{2,u}\beta^{1,d} & \beta^{2,u} \\ -y\alpha^{1,u} + \alpha^{2,d}\alpha^{1,d} & -y\beta^{1,u} + \alpha^{2,d}\beta^{1,d} & \beta^{2,d} \end{pmatrix} \times \begin{pmatrix} B \\ S^1 \\ S^2 \end{pmatrix}$$

3.1.4 Discrete model

To show that the risk-neutral probabilities can be interpreted as probabilities not only in the binary but also in the generic discrete case, the following **Markov property** of the Arrow-Debreu prices must be proved. Suppose that $A_{t,t+1}$ and $A_{t+1,t+2}$ denote the matrices of Arrow-Debreu prices for periods $[t, t+1]$ and $[t+1, t+2]$ (where $A_{t,s}^{i,j}$ is the price of the Arrow-Debreu security which pays $1 in state j in period s and zero otherwise conditional that in current period t the state is i). Then

$$A_{t,t+2} = A_{t,t+1} \times A_{t+1,t+2}$$

The Markov property is illustrated by the following example. Suppose that there are three periods in the binomial model and the states in period $t = 2$ are S^{uu}, S^{ud}, S^{du}, and S^{dd}. To replicate the payoff of the A^{uu} Arrow-Debreu security, investor applies the following strategy. If the price goes up in period $t = 1$, then investor purchases a replicating portfolio described in the previous section. If the price goes down, the investor gets zero payout and does nothing. The price of the replicating portfolio in state S^u is $A_{t+1}^{u,u}$. To generate the value in period $t + 1$ in state S^u, investor must purchase $A_{t+1}^{u,u}$ shares of the $\mathcal{A}^u$ security in period $t = 0$. The price of the $A_{t=1}^{u,u}$ shares is $A_{t=0}^{u}$. Therefore, the value of the $\mathcal{A}^{uu}$ security is $A_{t=2}^{uu} = A_{t=0}^{u} \times A_{t=1}^{u,u}$. The argument can be directly extended to prove the Markov property in general form.

Note that if the discrete process is modelled using a binary tree, then in general case the number of states in year T is $n = 2^{\frac{T}{dt}}$, where dt is the length of one period (in years). The Markov property allows to mitigate the problem of exponentially growing number of states as follows:

- The risk-neutral distribution in period $t = 1$ (denoted as Q_1) is approximated using the binomial tree model and the equations derived above. A tree step can be selected for example as $dt = 0.1$ so that $n = 2^{10}$ number of states in period $t = 1$ is sufficiently large to produce a good approximation of the risk-neutral probabilities but sufficiently small from the computational perspective;
- The risk-neutral distributions in periods $t = 2, \dots, T$ are calculated using the distribution Q_1 and the Markov property of the risk-neutral probabilities.

In the following sections, the risk-neutral probabilities are derived for different types of continuous processes (as a limit case of the binomial model with the tree step $dt \to 0$.

3.2 Black-Scholes model

A geometric Brownian motion model is applied to valuate stock price or commodity price options. A detailed discussion of the Bloomberg OV tool for the commodity option valuation is presented in **Error! Reference source not found.**. The risk-neutral probabilities are derived for a more general geometric Brownian motion model with mean reversion. The section shows that the risk-neutral probabilities do not depend on either drift or mean-reversion parameters μ and ρ.

3.2.1 Binomial approximation

The geometric Brownian motion is described by the following stochastic differential equation

$$S_{t+dt} = (S_t)^{1-\rho dt} \times e^{\mu dt + \sigma\sqrt{dt}\varepsilon_t} \tag{3.16}$$

where $\varepsilon_t \sim N(0,1)$ or in logarithm form

$$\Delta \ln S_{t+dt} = (\mu - \rho \ln S_t)dt + \sigma\sqrt{dt}\varepsilon_t \tag{3.17}$$

In each period, the process can be approximated by the following binary model:

$$\begin{cases} S^u = S^{1-\rho dt} \times e^{\mu dt + \sigma\sqrt{dt}} & p^u = \dfrac{1}{2} \\ S^d = S^{1-\rho dt} \times e^{\mu dt - \sigma\sqrt{dt}} & p^d = \dfrac{1}{2} \end{cases} \tag{3.18}$$

The equations can be approximated as follows:

$$\begin{cases} S^u = S^{1-\rho dt} \times \left[1 + \left(\mu + \dfrac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}\right] & p^u = \dfrac{1}{2} \\ S^d = S^{1-\rho dt} \times \left[1 + \left(\mu + \dfrac{\sigma^2}{2}\right)dt - \sigma\sqrt{dt}\right] & p^d = \dfrac{1}{2} \end{cases} \tag{3.19}$$

3.2.2 Continuous dividends

In the approximation, we assume that $R = r \times dt$ and $D = d \times dt$. The binary risk-neutral probabilities for the process are calculated using the following equations:[13]

[13] Specifically, $S^u - S^d = S \times \left(2\sigma\sqrt{dt}\right)$ and $q^u = \frac{S(1+(r-d)dt)-S^d}{(S^u-S^d)} = \frac{1}{2} + \frac{r-d-\mu-\frac{\sigma^2}{2}}{\sigma} \times \sqrt{dt}$.

(3.20)
$$\begin{cases} q^u = \frac{1}{2} + \frac{r - d - \mu - \frac{\sigma^2}{2} + \rho \ln S_t}{2\sigma} \times \sqrt{dt} \\ q^d = \frac{1}{2} - \frac{r - d - \mu - \frac{\sigma^2}{2} + \rho \ln S_t}{2\sigma} \times \sqrt{dt} \end{cases}$$

The process has the following risk-neutral mean and variance parameters:[14]

(3.21)
$$\begin{cases} E^Q[S] = S \times [1 + (r - d)dt] \\ \sigma^{2,Q}[S] = S^2 \times \sigma^2 \times dt \end{cases}$$

The equations illustrate the Girsanov theorem: if the stock price process described by the geometric Brownian motion with parameters (μ, σ), then the risk-neutral probabilities derived for the process are described by the geometric Brownian motion with parameters $\left(r - d - \frac{\sigma^2}{2}, \sigma\right)$. Note that the risk-neutral probabilities do not depend on the parameter μ. The risk-neutral probabilities are generated by replacing parameter μ with the following parameter:

(3.22)
$$\mu \to r - d - \frac{\sigma^2}{2}$$

3.3 Black-Scholes model with mean-reversion

A standard geometric Brownian motion model is applied to valuate stock price or commodity price options. A detailed discussion of the Bloomberg OV tool for the commodity option valuation is presented in **Error! Reference source not found.**.

3.3.1 Binomial approximation

The geometric Brownian motion is described by the following stochastic differential equation

(3.23)
$$S_{t+dt} = (S_t)^{1-\rho dt} \times e^{\mu dt + \sigma\sqrt{dt}\varepsilon_t}$$

where $\varepsilon_t \sim N(0,1)$ or in logarithm form

(3.24)
$$\Delta \ln S_{t+dt} = (\mu - \rho \ln S_t)dt + \sigma\sqrt{dt}\varepsilon_t$$

In each period, the process can be approximated by the following binary model:

---

[14] The risk-neutral variance equals the actual variance of the stock price process $\sigma^{2,Q}[S] = (S^2 2\sigma^2) \times \frac{1}{2} = S^2 \times \sigma^2 \times dt$. The equation is derived formally in Appendix B.1.

(3.25)
$$\begin{cases} S^u = S^{1-\rho dt} \times e^{\mu dt + \sigma\sqrt{dt}} & p^u = \dfrac{1}{2} \\ S^d = S^{1-\rho dt} \times e^{\mu dt - \sigma\sqrt{dt}} & p^d = \dfrac{1}{2} \end{cases}$$

The equations can be approximated as follows:

(3.26)
$$\begin{cases} S^u = S^{1-\rho dt} \times \left[1 + \left(\mu + \dfrac{\sigma^2}{2}\right) dt + \sigma\sqrt{dt}\right] & p^u = \dfrac{1}{2} \\ S^d = S^{1-\rho dt} \times \left[1 + \left(\mu + \dfrac{\sigma^2}{2}\right) dt - \sigma\sqrt{dt}\right] & p^d = \dfrac{1}{2} \end{cases}$$

3.3.2 Continuous dividends

In the approximation, we assume that $R = r \times dt$ and $D = d \times dt$. The binary risk-neutral probabilities for the process are calculated using the following equations:[15]

(3.27)
$$\begin{cases} q^u = \dfrac{1}{2} + \dfrac{r - d - \mu - \frac{\sigma^2}{2}}{\sigma} \times \sqrt{dt} \\ q^d = \dfrac{1}{2} - \dfrac{r - d - \mu - \frac{\sigma^2}{2}}{\sigma} \times \sqrt{dt} \end{cases}$$

The process has the following risk-neutral mean and variance parameters:[16]

(3.28)
$$\begin{cases} E^Q[S] = S \times [1 + (r - d)dt] \\ \sigma^{2,Q}[S] = S^2 \times \sigma^2 \times dt \end{cases}$$

The equations illustrate the Girsanov theorem: if the stock price process described by the geometric Brownian motion with parameters (μ, σ), then the risk-neutral probabilities derived for the process are described by the geometric Brownian motion with parameters $\left(r - d - \frac{\sigma^2}{2}, \sigma\right)$. Note that the risk-neutral probabilities do not depend on the parameter μ. The risk-neutral probabilities are generated by replacing parameter μ with the following parameter:

(3.29)
$$\mu \to r - d - \frac{\sigma^2}{2}$$

[15] Specifically, $S^u - S^d = S^{1-\rho dt} \times \left(2\sigma\sqrt{dt}\right)$ and $q^u = \frac{S(1+(r-d)dt) - S^d}{(S^u - S^d)} = \frac{1}{2} + \frac{r - d - \mu - \frac{\sigma^2}{2}}{\sigma} \times \sqrt{dt}$.

[16] The risk-neutral variance equals the actual variance of the stock price process $\sigma^{2,Q}[S] =$.

3.4 CDS model

The credit default swap (**CDS**) valuation is an example of a binary continuous model with an absorbing state which corresponds to the default on the reference instrument. The absorbing state corresponds to the loan default state. A more detailed discussion of the CDS valuation is provided in the ‘Financial Guarantees’ guide.

The model assumes deterministic movement in the underlying asset price with probability close to one and a small probability γdt of jumping into the absorbing state X. After moving into state X, the process stays in the state X with probability one.

An example of the model is the price of a bond which is valued at par (S_t=100) but with probability γdt the bond may default in each period. Default is assumed to be the absorbing state and the value X of the bond in the state is interpreted as the bond recovery value. The model is described using $F(S)$=0.

The model is described by the following stochastic differential equation:

(3.30)
$$\begin{cases} dS_t = F(S_t) \times dt & p = 1 - \gamma dt \\ S_{t+dt} = X & p = \gamma dt \end{cases}$$

The model is described by a binomial process, which is equivalently described as follow:

(3.31)
$$\begin{cases} S^u = S + F(S) \times dt & p = 1 - \gamma dt \\ S^d = X & p = \gamma dt \end{cases}$$

The risk-neutral probabilities for the process are calculated using the following equations:[17]

(3.32)
$$\begin{cases} q^u = 1 - \left(\dfrac{F(S)}{S-X} + \dfrac{S}{S-X}(d-r) \right) \times dt \\ q^d = \left(\dfrac{F(S)}{S-X} + \dfrac{S}{S-X}(d-r) \right) \times dt \end{cases}$$

The process has the following risk-neutral mean and variance parameters:

(3.33)
$$\begin{cases} E^Q[S] = S \times [1 + (r-d)dt] \\ \sigma^{2,Q}[S] = S(S-X) \times \left[\dfrac{F(S)}{S} + d - r \right] \times dt \end{cases}$$

The parameters of the risk-neutral probabilities do not depend on default hazard rate parameter γ. The risk-neutral probabilities are generated by replacing parameter γ with the following parameter:

---

[17] Specifically, $S^u - S^d = S - X + F(S)dt$ and $q^u = \frac{S(1+(r-d)dt)-S^d}{(S^u-S^d)} = \frac{S-X+S(r-d)dt}{S-X+F(S)dt} = 1 - \left(\frac{F(S)}{S-X} + \frac{S}{S-X}(d-r)\right) \times dt$.

(3.34) $$\gamma \to \frac{S}{S-X} \times \left[\frac{F(S)}{S} + d - r\right]$$

A special case of the formula in the case of non-amortizable bond is presented in equation (3.35) of Section 3.5.4.

3.5 Bond valuation models

3.5.1 Model of bond prices

The Vasicek model is specified using the following stochastic model of interest rates:

$$dr_t = (\vartheta - ar_t) \times dt + \sigma \times dW_t = \vartheta \times dt + \sigma \times dW_t \quad (as\ a \to 0)$$

For simplicity of presentation, the analysis in this guide is limited to the case $a = 0$. Zero coupon prices are described by the following equation:

$$P_{t,T} = A_{t,T} \times e^{-B_{t,T} \times r}$$

where

$$B_{t,T} = T - t$$

and

$$A_{t,T} = e^{-\left[\vartheta \frac{(T-t)^2}{2} - \frac{\sigma^2 (T-t)^3}{6}\right]}$$

The stochastic model of the bond price is described by the following equation:[18]

$$\begin{aligned} dP &= \frac{\partial P}{\partial t} \times dt + \frac{\partial P}{\partial r} dr + \frac{1}{2} \times \frac{\partial^2 P}{\partial r^2} \times \sigma^2 \times dt \\ &= \left(\frac{A'}{A} - B'r\right) \times P \times dt - B \times P \times [\vartheta \times dt + \sigma \times dW_t] + \frac{1}{2} \times B^2 \times \sigma^2 \times P \times dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} dP &= \left[\vartheta \times (T-t) - \frac{\sigma^2 (T-t)^2}{2} + r - \vartheta(T-t) + \frac{1}{2} \times (T-t)^2 \times \sigma^2\right] \times P \times dt - B \times \sigma \times P \times dW \\ &= r \times P \times dt - B \times \sigma \times P \times dW \end{aligned}$$

The stochastic equation for the bond price is simplified as follows:

$$\begin{cases} d \ln P &= r \times dt - \sigma(T-t) \times dW \\ dr &= \vartheta \times dt + \sigma \times dW_t \end{cases}$$

[18] Specifically, $B' = -1$ and $\frac{A'}{A} = \vartheta \times (T-t) - \frac{\sigma^2 (T-t)^2}{2}$.

The stochastic equation is modelled as a two-dimensional system in which the variables P and r are linked as follows:

$$\ln P_{t,T} = -\left[\vartheta \frac{(T-t)^2}{2} - \frac{\sigma^2 (T-t)^3}{6}\right] - (T-t) \times r_t$$

The stochastic changes in the market interest rates and bond prices are calculated simultaneously.

3.5.2 Risk-neutral prices

3.5.3 Black model

The Black model is illustrated for the case of Vasicek model of interest rates. The risk-neutral probabilities are derived in two steps. First, the model of bond price (underlying asset) diffusion process is derived. Next, the binomial approximation to the bond price and respective risk-neutral probabilities are derived.

The risk-neutral probabilities are described by the following equations:

$$\begin{cases} S^u = S \times \left[1 + \left(r + \frac{\sigma^2 (T-t)^2}{2}\right) dt + \sigma (T-t) \sqrt{\frac{dt}{2}}\right] & p^u = \frac{1}{2} \\ S^d = S \times \left[1 + \left(r + \frac{\sigma^2 (T-t)^2}{2}\right) dt - \sigma (T-t) \sqrt{\frac{dt}{2}}\right] & p^d = \frac{1}{2} \end{cases}$$

and

$$\begin{cases} q^u = \frac{1}{2} - \frac{\sigma (T-t)}{2} \times \sqrt{\frac{dt}{2}} \\ q^d = \frac{1}{2} + \frac{\sigma (T-t)}{2} \times \sqrt{\frac{dt}{2}} \end{cases}$$

3.5.4 Bond CDS model

Suppose that the binary process models the bond price with $S = 100$ and $F(S) = -\lambda S_0$ (straight line loan amortization over the loan maturity period. If $\lambda = 0$, then the bond is priced at par in each period). In the event of bond default, the bond residual value equals $X = \alpha S$, where α is the bond recovery rate at default. The model dividend payment equals to the bond principal amortization and interest expense amounts. Dividend payout in each period equals $D^{fixed} = \lambda S_0 \times dt + i \times S \times dt$. Therefore, $d = \frac{D^{fixed}}{S \times dt} = \frac{\lambda S_0}{S} + i$). After we substitute the equations into the equation (3.34), the expression for the **risk-neutral default hazard rate** is described by the following equation:

(3.35) $$\gamma \to \frac{i - r}{1 - \alpha}$$

The actual default hazard rate γ equals the risk-neutral default hazard rate whenever

(3.36) $$i = r + \gamma \times (1 - \alpha) = r + \pi$$

where $\pi = \gamma \times (1 - \alpha)$ is interpreted as recovery-adjusted risk premium.

3.6 Lease residual value risk model

Lease residual risk valuation is modelled in this guide either as (i) a Black-Scholes model or (ii) a CDS model. Under the Black-Scholes modelling approach, the residual value of the lease underlying asset is assumed to change continuously and depend on the market, technological or regulatory risks. Under the CDS modelling approach, the residual value is modelled as a binary process where the 'default' state corresponds to destruction of the underlying asset value.

Leasing business involves operational costs which must be taken into consideration in the lease derivative pricing. Specifically, the dividends parameter in the Black-Scholes or CDS model is estimated as the lease fee revenues adjusted by the operating costs:

(3.37) $$D = F \times m^{op}$$

where F is the lease fee and m^{op} is the operating margin of the lessor. We refer to the parameter D in the lease residual value risk model as a dividend payment (to show the relation to a CDS / Black-Scholes model) or cost-adjusted lease fee payment.

Most lease contracts specify monthly lease fee payment frequency. Therefore, for consistency we assume that

$$dt = \frac{1}{12}$$

throughout these notes whenever we discuss the lease residual valuation models.

3.6.1 Black-Scholes approach

Under the Black-Scholes approach, the residual value is modelled using geometric Brownian motion, which is described by equation (3.16). The risk-neutral probabilities are described by the same equation with parameter μ replaced with

(3.38) $$\mu \to r - d - \frac{\sigma^2}{2}$$

where the dividend rate is calculated using equation (4.7). The dividend rate includes the market depreciation of the asset and the interest component:

(3.39) $$d = \lambda + i$$

Therefore, the above equation can be equivalently represented as follows:

(3.40) $$\mu \rightarrow= -\left(\lambda + (i - r) + \frac{\sigma^2}{2}\right)$$

The residual value of the asset under the risk-neutral probabilities is described by the following stochastic equation:

(3.41) $$\ln S_t = \ln S_0 - \left(\lambda + (i - r) + \frac{\sigma^2}{2}\right) t + \sigma \varepsilon_t$$

Parameter

(3.42) $$\tilde{\lambda} = \lambda + (i - r) + \frac{\sigma^2}{2}$$

is interpreted as the (negative) slope of the **implied premium schedule**, which includes the following components: (i) compensation for asset depreciation (estimated based on market depreciation schedule); (ii) risk premium component of the lease interest rate (estimated as the lease interest rate and risk-free interest rate); and (iii) residual value volatility parameter σ.

Note that the implied depreciation schedule is derived from the lease fees and risk-free rates and does not depend on the actual depreciation schedule estimated using historical data. Equation (3.42), which compares the implied and actual depreciation schedule slopes, is presented in this format for convenience to break down the impact of the expected actual depreciation (λ) and risk premium ($\pi = (i - r) + \frac{\sigma^2}{2}$) components on the option price.

3.6.2 CDS approach

In this section, we show that under the CDS approach the lease instrument can be interpreted as an amortized loan. The lease residual value is priced similar to a CDS price of the amortized loan. Note that the risk-neutral hazard rate for a generic CDS model is described by the equation (3.34).

(3.43) $$\gamma \rightarrow \frac{S}{S - X} \times \left[\frac{F(S)}{S} + d - r\right]$$

We consider two alternative residual value model specifications.

3.6.2.1 Continuous pricing

"Continuous pricing" model is presented for modelling purposes. In practice, a lease fair market value (**FMV**) is observed only in a single period when the lease is issued. The equations presented in this section are applied to derive the valuation process in the "single-price" model.

Under the "continuous pricing" model, we assume that the market lease fee is re-values in each period and is quoted as a sum of the expected depreciation and lease interest rate. The lease interest rate is calculated as a fixed percentage of the lease residual market value:

$$d \times S \times dt = -F(S)dt + i \times S \times dt$$

The implied default hazard rate is calculated consistently with equation (3.35) as

$$\gamma = \frac{\frac{F(S)}{S} + d - r}{1 - \alpha} = \frac{i - r}{1 - \alpha} \tag{3.44}$$

Under the "continuous pricing" CDS approach, the lease residual risk model is interpreted as the amortized bond CDS model. Specifically, (i) lease depreciation schedule corresponds to the bond amortization schedule; (ii) lease interest rate (i) corresponds to the bond interest rate; and (iii) lease value recovery rate in the event of default (α) corresponds to the bond value recovery rate.

3.6.2.2 **Single issue date pricing**

3.6.2.3 **Counter-party risk**

Note that under the CDS valuation approach, the return on the lease contracts (adjusted for residual value risk and operational costs) is equal to the risk-free rate. Suppose that the derivative contracts are issued by two affiliated entities within the same corporate group. The approach described in the sections above does not take into account the risk of default by the entity which performs the functions of insurer in the group.[19]

To account for the counter-party default risk, the risk-free rate of return is replaced with the rate of return applicable to the insurer. The group rate of return can be used as a proxy to replace the risk-free rate.

$$r \to i^{group}$$

And the implied hazard rate is estimated as

$$\gamma = \frac{i - i^{group}}{1 - \alpha}$$

3.6.3 Insurance models of residual risk

The equations derived above are arbitrage-free prices, which incorporate both the expected costs and profit components. Specifically, under the Black-Scholes modelling approach, the expected costs are compensated through the expected depreciation compensation component of the implied premium schedule and the profit component is priced through the differential between the lease interest rate and risk-free rate ($i - r$ component) and through the residual value volatility component ($\frac{\sigma^2}{2}$).

[19] The minimum rate at which the group can raise capital in the markets is the refinancing rate of return. Therefore, the group refinancing rate of return must be retained by the insured entity.

Under the CDS modelling approach, the lease residual value risk model is interpreted as a CDS model of an amortized bond asset. The arbitrage-free price is derived based on the implied default hazard rate of the underlying bond asset.

Under the insurance approach, the implied parameters in the price equations (estimated based on market prices) are replaced with the parameters based on historical data. The insurance approach effectively applies the same equations to price the derivatives but uses a different approach for equation parameters estimation. Implementation of the insurance approach for the continuous (Black-Scholes) and binary (CDS) models is described as follows:

Continuous model

Binary (default / non-default) model

3.7 Loan guarantee models

This section derived risk-neutral probabilities for the downstream and upstream loan guarantee models.

3.7.1 Downstream loan guarantee

3.7.2 Upstream loan guarantee

3.8 Hierarchical structures

The models discussed in this section are three-asset models including a risk-free asset and two risky assets. The price movement of the risky assets can be represented by an hierarchical tree structure: (i) first, a binomial tree models the movement in the ‘parent’ asset; (ii) then, conditional on the movement of the ‘parent’ asset price, a binomial movement in the ‘child’ asset is modelled.

3.8.1 Black-Scholes model with market index

3.8.2 Share purchase commitment

Section 4 Derivative Prices

In this section, the derived risk-neutral probabilities are applied to calculate the derivative prices.

4.1 General model specifications

The derivative prices are first discussed within the context of general model specifications and then applied to different financial instruments.

4.1.1 Discrete model

4.1.2 Continuous model

The option price is calculated as the expectation of the option payoff under the risk-neutral probability distribution:

$$V = e^{-rT} \times E^Q[F(S)] \quad (4.1)$$

The payoff function of the stock price call option is defined as follows:

$$F^{call}(S) = (S - K)^+ \quad (4.2)$$

where K is the strike price and function x^+ is defined as $x^+ = \max[x, 0]$. The payoff function of the stock price put option is defined as follows:

$$F^{put}(S) = (K - S)^+ \quad (4.3)$$

The expectations are taken with respect to risk-neutral probabilities described by the following equation

$$S_{t+dt} = S_t \times e^{\left(r-d-\frac{\sigma^2}{2}\right)dt+\sigma\sqrt{dt}\varepsilon_t} = S_0 e^{\left(r-d-\frac{\sigma^2}{2}\right)\times(t+dt)+\sigma\sqrt{t+dt}\times\varepsilon}$$

As an example, suppose that $F(S) = S$. Then

$$E^Q[S_t] = S_0 \times e^{\left(r-d-\frac{\sigma^2}{2}\right)\times t} \times e^{\frac{\sigma^2 t}{2}} = S_0 e^{(r-d)t}$$

4.2 Black-Scholes formula

Standard and alternative formula presentation and formula derivation are presented below.

4.2.1 Forward price

The payoff function in the forward contract is described as $F(S) = S - K$ and the forward prices is described respectively as

$$V = e^{-rT} \times E^Q[S - K]$$

In the Black-Scholes case, the equation is represented respectively as follows

(4.4) $$V^{fwd} = e^{-rT} \times S \times e^{(r-d)T} - Ke^{-rT} = S \times e^{-dT} - Ke^{-rT}$$

At the issued date the price K is set so that $V^{fwd} = 0$ or respectively

(4.5) $$K = S_0 \times e^{(r-d)T} = E^Q[S_T]$$

After the issue date the price of a forward contract deviates from the original zero price and is described by the following equation:

(4.6) $$V^{fwd} = e^{-rT} \times S \times e^{(r-d)T} = (S - S_0) \times e^{-dT}$$

4.2.2 Dividends

This section derives the price for a fixed dividend payment $F(S) = D$ and the price for continuous dividend cash flow $F(S_t) = d \times S_t$. The equation for the fixed dividends is straightforward:

$$V(D) = D \times e^{-rt}$$

The value of the continuous dividend cash flow is described as follows

$$V = \int_0^t d \times e^{-rt} \times E^Q(S_t)dt = d \times t \times S_0$$

The conversion of the fixed discrete dividends sequence into the implied parameter d is described by the following equation:

$$d = \frac{D_0 \times \sum_i e^{-rt_i}}{T \times S_0}$$

In the case of regular fixed dividend payment frequency (e.g. monthly frequency), the above equation can be simplified as follows:

(4.7) $$d = \frac{D_0}{T \times S_0} \times \frac{1 - e^{-r \times (n+1)/12}}{1 - e^{-r/12}}$$

where n is the number of fixed dividend payments over the contract term T. (The equation can be directly modified for other payment frequencies).

4.2.3 Call / Put price

This section provides a standard (Wikipedia) presentation of the Black-Scholes formula and then shows how the formula is derived from the risk-neutral probabilities described in the previous section.

4.2.3.1 Standard presentation

Standard Black-Scholes option prices are described by the following equations.[20]

Call option

(4.8)
$$V^{call} = N(d_1) \times Se^{-dT} - N(d_2) \times Ke^{-rT}$$

Put option

(4.9)
$$V^{put} = -N(-d_1) \times Se^{-dT} + N(-d_2) \times Ke^{-rT}$$

where parameters d_1 and d_2 are estimated as

(4.10)
$$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}} \times \left[\ln\frac{S}{K} + \left(r - d + \frac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

4.2.3.2 Alternative representation

Alternative representation of the Black-Scholes formula can be represented as follows. The forward price in Black-Scholes model is described by the following equation:

$$F = S \times e^{(r-d)T}$$

Replacing spot price with the forward price, the call and put option prices can be described by the following formulas.

Call option

(4.11)
$$V^{call} = e^{-rT} \times [N(d_1) \times F - N(d_2) \times K]$$

Put option

(4.12)
$$V^{put} = e^{-rT} \times [-N(-d_1) \times F + N(-d_2) \times K]$$

Where parameters d_1 and d_2 are estimated as

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[20] https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model

(4.13)
$$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}} \times \left[\ln\frac{F}{K} + \frac{\sigma^2}{2}T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

4.2.3.3 **Non-linear variance of cumulative residuals**

Under the geometric Brownian motion assumption, the cumulative residuals term is equal to the sum of independently distributed period-specific terms with variance parameter equal to $\sigma^2 \times dt$. The variance parameter of the cumulative residual term equals to $\sigma^2 \times T$ (the variance term increases linearly with the derivative contract maturity term).

In certain cases, empirical evidence does not support the assumption of linear relationship between the maturity term and variance of the cumulative residual term. Therefore, the above equations need to be generalized for a generic functional form $\sigma(T)$ of the cumulative residual term. The generalized equations are presented as follows:

$$d_1 = \frac{1}{\sigma(T)} \times \left[\ln\frac{S}{K} + (r-d) \times T + \frac{\sigma^2(T)}{2}\right]$$

And

$$d_2 = d_1 - \sigma(T)$$

The equations (4.11) and (4.12) for the call/put prices remain the same with the modified equations for the parameters d_1 and d_2. [NTD: Need to show it formally]

4.2.3.4 **Formula derivation**

We illustrate below how the equations are derived as the risk-neutral expectations of the call and put payoff functions. Under the geometric Brownian motion, the distribution of the stock price S_T is described by the following equation:

$$S_T = S_0 \times e^{\mu T + \sigma\sqrt{T}\varepsilon_T}$$

where $\varepsilon_T \sim N(0,1)$. To calculate the option prices, the expectations are calculated for parameters (μ, σ) and then the parameter μ is replaced with the equivalent risk-neutral parameter $\mu \to r - d - \frac{\sigma^2}{2}$.[21]

$$V^{call} = e^{-rT} \times \frac{1}{\sqrt{2\pi T}\sigma} \times \int_{\ln\frac{K}{S}}^{\infty} (Se^x - K) \times e^{-\frac{(x-\mu T)^2}{2\sigma^2 T}} dx = e^{-rT} \times S \times e^{\mu T + \frac{\sigma^2 T}{2}} \times N\left(-\frac{\ln\frac{K}{S} - \mu T - \sigma^2 T}{\sigma\sqrt{T}}\right) - e^{-rT} \times K \times N\left(-\frac{\ln\frac{K}{S} + \mu T}{\sigma\sqrt{T}}\right)$$

[21] The Black-Scholes formula is derived for the call option only. The calculations presented in the notes can be extended directly to the Black-Scholes formula for the put option.

After replacing $\mu \to r - d - \frac{\sigma^2}{2}$, the above equation is simplified as follows:

$$e^{-rT} \times e^{\mu T + \frac{\sigma^2 T}{2}} = e^{-dT}$$

and

$$-\frac{\ln\frac{K}{S} + \mu T + \sigma^2 T}{\sigma\sqrt{T}} = \frac{\ln\frac{S}{K} + \left(r - d + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1$$

Therefore, the discount expected risk-neutral value of the call payoff function equals exactly to the equation described by the Black-Scholes formula.

4.3 CDS

Suppose that K is the compensation at default (strike price); $X = \alpha S$ is recovery at default state; and $\frac{F(S)}{S} = -\lambda$ are a constant. Then the probability of the process moving to the absorbing state over the period T equals to $P^X = 1 - e^{-\gamma T}$ and the discounted expected payoff is equal to

$$V = (K - X) \times \int_0^T \gamma e^{-rt} e^{-\gamma t} dt$$

For a fixed value of γ, the equation can be simplified as follows.

$$V = (K - X) \times \frac{\gamma}{\gamma + r} \times \left(1 - e^{-(\gamma + r)T}\right)$$

4.3.1 Zero-coupon bonds

Zero-coupon bond is a transaction which pays fixed \$1 amount at a specific period t in the future conditional that the bond is not in the default state. Therefore, it can be viewed as similar to a forward contract in Black-Scholes model. The price of zero-coupon bond is described by the following equation:

$$V^{zc} = e^{-(\gamma + r)T} = e^{-iT}$$

where $\gamma = \frac{i-r}{1-\alpha} = i - r$ (and $\alpha = 0$). Prices of zero-coupon bonds are applied to price fixed payments which are conditional on the non-default state of the underlying asset.

4.3.2 CDS price

CDS price is derived in this section assuming that dividends are paid continuously and are specified as a fixed share of current asset price. Conversion of discrete into continuous dividends is discussed in Section 0.

$$D = d \times S$$

The value of put option is obtained when the actual hazard rate γ is replaced with the risk-neutral hazard rate: $\gamma \to \frac{S}{S-X} \times \left[\frac{F(S)}{S} + d - r\right]$.

In the case when the put option represents the CDS contract on the bond transaction,

$$\gamma = \frac{(d-\lambda) - r}{1-\alpha} = \frac{i-r}{1-\alpha}$$

where $i = d - \lambda$, $K = kS$, and $K - X = (k-\alpha)S$. The above equation can be represented then equivalently as follows:

(4.14) $$V = S(i-r) \times \frac{k-\alpha}{1-\alpha} \times \int_0^T e^{-rt} e^{-\gamma t} dt$$

where $e^{-\gamma t}$ represents the probability that the bond default does not default prior to period t. The equation is interpreted as follows: the CDS seller receives the periodic payment at the rate equal to $(i-r) \times \frac{k-\alpha}{1-\alpha}$ conditional on the fact that the bond is not in the default state. (Specifically, $S(i-r)\left(\frac{k-\alpha}{1-\alpha}\right)dt$ term represents periodic CDS payment in period t, e^{-rt} represents risk-free discount factor, and $e^{-\gamma t}$ represents the probability that the bond is in the non-default state).

The interpretation is valid whenever the bond is priced consistently with the bond default hazard rate ($i = r + \gamma \times (1-\alpha)$). Otherwise, the above equation shows that risk-neutral non-default probability $e^{-\frac{d-r}{1-\alpha}t}$ must be used in the CDS value calculations. The value

(4.15) $$\tilde{\gamma} = \frac{i-r}{1-\alpha}$$

can be interpreted as **implied** bond default hazard rate.

The term

(4.16) $$AAF = \int_0^T e^{-rt} e^{-\gamma t} dt$$

is interpreted as expected annuity adjustment factor (**AAF**), which is defined as the discounted value of the sum of periodic fixed $1 payments conditional on the non-default state of the underlying asset. The CDS price is represented as

$$V = S(i-r) \times \frac{k-\alpha}{1-\alpha} \times AAF$$

and equivalent periodic fee is described by the following equation

$$v = \frac{V}{AAF} = S(i-r) \times \frac{k-\alpha}{1-\alpha}$$

Alternatively, CDS price can be quoted as a percentage of the notional amount S, which is paid periodically to the CDS seller:

(4.17)
$$f = (i - r) \times \frac{k - \alpha}{1 - \alpha} = \tilde{\gamma} \times (k - \alpha)$$

This is a standard representation of the CDS price (see the "Financial Guarantee" guide which contains a more detailed discussion of the CDS valuation models). If $k = 1$, the equation becomes a standard CDS price equation

(4.18)
$$f = i - r$$

4.3.3 Intercompany financial guarantees

A financial guarantee of a loan transaction can be viewed as a CDS instrument, which price is adjusted for the counterparty (guarantor) risk. Three assets are represented by three bonds: (i) a risk-free bond; (ii) a bond issued by the parent entity; and (iii) the bond issued by the subsidiary entity. The binomial tree is presented as the movement in the 'parent' bond and 'child' bond prices, where the 'parent' bond is the bond issued by the parent entity and the 'child' bond is issued by the subsidiary entity.

The two absorbing states correspond to the borrower and guarantor default states. A more detailed discussion of the loan guarantee valuation is provided in the 'Financial Guarantees' guide.

4.4 Bonds

4.4.1 Black formula

4.4.2 Bond CDS valuation

4.4.3 Convertible bonds

In this section we present the results of convertible bond valuation performed for the bond illustrated in Section 2.6.1. The valuation is performed using Black-Scholes model. In Appendix **Error! Reference source not found.** the results of Black-Scholes model are compared to the results of Bloomberg OVCV tool.

The option valuation tool uses the following input parameters:

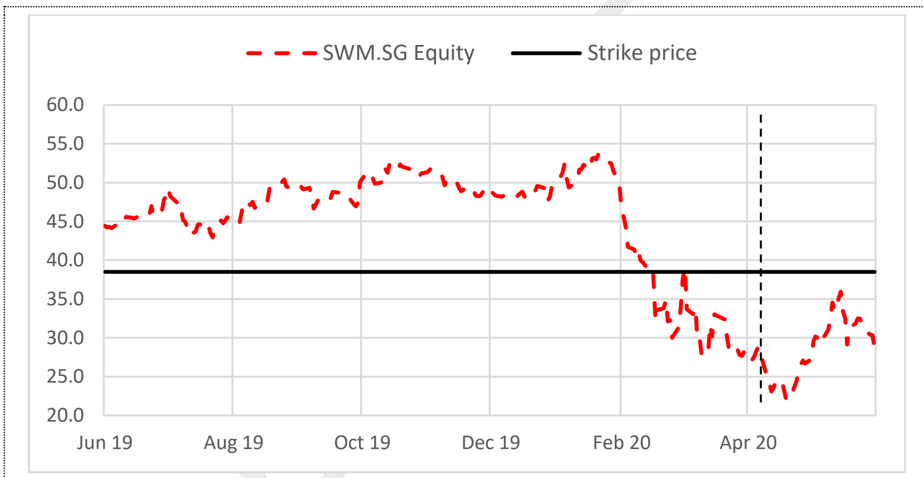
1. Terms of the bond transaction: maturity date, coupon rate, coupon frequency and day count

2. Terms of conversion option: stock conversion price, last conversion date (typically equals to maturity date minus one day), valuation date of the option valuation analysis
3. Bond market price as of the valuation date
4. Stock historical prices and price as of the valuation date
5. Risk-free rate

Application of the convertible bond valuation is illustrated for the bond which terms are summarized in Bloomberg print screen shown in Section 2.6.1. The bond was issued by Southwest Airlines Co. at the end of April 2020 in the mid of COVID-19 economic crisis. The convertible bond allowed to raise the funds at a low cost (fixed 1.25% coupon rate). The value of the bond was primarily driven by the expectations that the COVID-19 crisis is temporary, the stock price of the company will recover and the high return on the bond will be generated through conversion of bond into the company shares.

The historical share price, strike price set in the convertible bond terms, and bond issue date are illustrated in the exhibit below.

Exhibit 4.1 Movement in the historical share prices and convertible bond strike price



Technically, the conversion option can be exercised at any time prior to the last conversion date and, therefore, the option is similar to American call option. However, for simplicity we assume that the option is exercised exactly on the last conversion date and, therefore, is similar to the European call option.

The value of bond convertibility option can be represented either as (i) change in the bond price; or as (ii) change in the bond yield. The steps of option valuation are summarized as follows.

1. Estimate stock price volatility (denoted as σ) based on the historical movement in the share prices.
2. Estimate the call option value (denoted as C^{share}) for each traded share using Black-Scholes formula (4.8) for European call option.

$$C^{share} = N(d_1) \times Se^{-dT} - N(d_2) \times Ke^{-rT}$$

where

$$\begin{cases} d_1 = \dfrac{1}{\sigma\sqrt{T}} \times \left[\ln\dfrac{S}{K} + \left(r - d + \dfrac{\sigma^2}{2}\right)T\right] \\ d_2 = d_1 - \sigma\sqrt{T} \end{cases}$$

3. Multiply call option price C^{share} by the conversion ratio to estimate the value of call option per bond par value (denoted as C^{par}). The value C^{par} represents the price of the convertibility option represented in terms of bond price change.
4. For convertible option estimated based on bond market price P^{FMV}, proceed with the following steps:
   - Subtract the value C^{par} from the bond price P^{FMV} to estimate the bond floor price P^{floor}.
   - Estimate the yield rate y^{FMV} based on bond market price P^{FMV}.
   - Estimate the bullet bond yield rate y^{bullet} based on the price P^{floor}.
   - Estimate bond yield adjustment as $\Delta y = y^{bullet} - y^{FMV}$. The yield Δy represents the bond yield rate adjusted for the convertibility option.
5. For convertible option estimated based on bond credit spread, proceed with the following steps
   - Estimate bullet bond yield ratey^{bullet} as risk-free rate plus credit spread.
   - Estimate bond floor price based on bond coupon rate and discount rate y^{bullet}.
   - Estimate bond FMV as bond floor price plus C^{par}.
   - Estimate bond yield rate y^{FMV} based on bond FMV value.
   - Estimate bond yield adjustment as $\Delta y = y^{bullet} - y^{FMV}$.

The second approach described in item 5 is the default approach in Bloomberg OVCV option valuation tool (see Appendix **Error! Reference source not found.** for further details). The approach described in item 4 can be produced by the OVCV tool if the credit spread parameter is replaced with the implied credit spread value. [Check]

Application of the option valuation steps are illustrated for the convertible bond issued by Southwest Airlines Co. All calculations are performed per 100 par value of the bond.

1. Volatility σ was estimated at 61.6% using a 1-year sample daily share price data and applying the following formula
2. The value C^{share} was estimated at 16.28 based on equation (4.8).
3. The value C^{par} was estimated as $C^{par} = 2.6 \times 16.28 = 42.31$.
4. For convertible option estimated based on bond market price $P^{FMV} = 121.73$, the yield adjustment was estimated as follows:
   - The bond floor price P^{floor} was estimated at 79.42.
   - The yield rate y^{FMV} was estimated at -2.88%
   - The yield rate y^{bullet} was estimated at 6.22%.
   - The yield adjustment was estimated at $\Delta y = 9.11\%$.
5. For convertible option estimated based on bond credit spread of 1.93%[22], the yield adjustment was estimated as follows.
   - Bullet bond yield ratey^{bullet} was estimated at 2.33%.[23]

[22] The credit spread was selected based on the OVCV tool (see Appendix **Error! Reference source not found.**). Alternatively, credit spread can be estimated based on interest benchmarking analysis.

[23] Risk-free rate was estimated at 0.4% based on Libor swap curves.

- ► Bond floor price was estimated at 95.12.
- ► Bond FMV price was estimated at 137.44.
- ► The y^{FMV} yield rate was estimated at -5.39;
- ► The yield adjustment was estimated at $\Delta y = 7.72\%$.

Adjustment for convertibility, estimated in item 4 above shows that if the bond was issued as a bullet bond, it would be priced at a significant discount (6.22% compared to the 1.25% coupon rate and 2.33% market discount rate). Including the convertibility option into the terms of the bond mitigates the high cost of financing.

The following data is required to perform the valuation of a convertible bond:

1. Terms of the convertible bond (obtained through Bloomberg database)
   - ► Last conversion date (or maturity date if same as last conversion date), T
   - ► Stock conversion price, K
2. Stock price to estimate the stock price volatility (obtained through Bloomberg or yahoo.finance[24])
3. Convertible bond market price
4. Risk-free rate (estimated using US$ Libor swap rates, overnight interest swap (OIS) curve, or US treasury yield rates (obtained through Bloomberg database).

4.5 Leases

In the case of lease contracts, the option pricing theory is applied to estimate the price of the residual value risk. This section presents three alternative approaches to price residual value risk.

4.5.1 Black-Scholes approach

4.5.2 CDS approach

4.5.3 Mixed approach

4.6 Trader approach to derivative pricing

[24] https://finance.yahoo.com/quote/NAV/history?p=NAV

Section 5 Arbitrage vs Insurance Pricing

This section compares the arbitrage and insurance approaches to derivative pricing and illustrates the application of the two approaches for different financial models.

5.1 CDS

In this section, we

5.1.1 Relation between insurance and arbitrage prices

Under the insurance interpretation of the CDS contract, the periodic compensation to the CDS seller is estimated as follows. The value of the CDS contract is assessed based on the **expected loss** to the CDS seller. The CDS periodic fee is set such that the CDS seller's expected cost equals to the **expected benefit**.

Suppose that γ denotes default hazard rate on the underlying bond estimated based on the analysis of comparable bonds historical default data. The expected loss (denoted as EL) to the CDS seller is estimated as follows:

$$EL = (k - \alpha) \times S \times \int_0^T \gamma \times e^{-\gamma t} \times e^{-rt} \times dt$$

where $\gamma \times e^{-\gamma t}$ is the pdf function of the bond default event and e^{-rt} is risk-free discount factor. Assuming constant parameter γ, the equation can be represented equivalently as

$$EL = \gamma \times (k - \alpha) \times S \times AAF$$

The expected benefit to the CDS seller (denoted as EB) is estimated as

$$EB = f \times S \times AAF$$

The fee f is estimated from the equation $EL = EB$:

(5.1) $$f = \gamma \times (k - \alpha)$$

Under the arbitrage pricing approach, the CDS periodic fee was derived in Section 4.3.2 as

(5.2) $$f = \tilde{\gamma} \times (k - \alpha)$$

where

(5.3) $$\tilde{\gamma} = \frac{i - r}{1 - \alpha}$$

Based on the considerations above, the comparison between the insurance and arbitrage pricing approaches can be summarized as the following statement.

Lemma. Under both the insurance and arbitrage pricing approaches, the CDS fee is estimated using the same equation (5.1) and (5.2) but by applying different estimation approach an interpretation of parameter γ:

1. Under the insurance approach, parameter γ is interpreted as default hazard rate and is estimated based on historical default data of comparable bond transactions;
2. Under the arbitrage approach, parameter $\tilde{\gamma}$ is interpreted as implied default hazard rate and is derived from the bond market price using equation (5.3). The implied hazard rate equals to the risk premium component ($\pi = i - r$) of the bond interest rate adjusted for the defaulted bond recovery rate.

5.1.2 Application of insurance and arbitrage pricing

Application of the arbitrage and insurance pricing is illustrated using the FY2020 market yield data for bonds issued in US$ Industrial sector and Moody's tables with historical bond default rates.

5.2 Leases

Section 6 Parameter Estimation

This section described parameter estimation methodologies for the described derivative models. The data used in the derivative pricing can be presented in different forms.

1. A derivative for a single instrument is priced. The price information for the underlying instrument is available as a time-series data. This case is applicable for example when a derivative for an underlying stock traded in the market is priced.
2. Multiple derivatives for multiple instruments are priced. The price information for the underlying instruments is available as a time-series data. This case is applicable for example when a portfolio of derivatives for underlying stocks traded in the market are priced.
3. Multiple derivatives for multiple instruments are priced. The price information for the underlying instruments is available for a single period of time. This case is applicable for example when a derivative for lease agreement is priced. Lease and underlying asset prices are available only at the lease agreement date

Cases 2 and 3 are analyzed as a collection of single instruments. However, estimation of some parameters, such as for example volatility, is based on the portfolio data which may be aggregated and screened for outliers.

6.1 Black-Scholes model

Under the Black-Scholes model, the following list of parameters is estimated.

1. Risk-free rate r
2. Dividend rate d
3. Volatility σ and drift parameter λ.[25]

6.1.1 Risk-free rate

Risk-free rate is estimated based on various yield series, such as

1. US$ Treasury rates
2. Libor swap curve
3. Overnight index swap (OIS) curve[26]

The curves can be obtained through Bloomberg database.[27] A more detailed discussion of the risk-free rate is provided in XXX.

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[25] Note that while parameter λ is not used directly in the derivative price equation, its estimation is required for volatility parameter estimation.

[26] The industry practice of applying discount rates based on the Libor and OIS rates is discussed in J. Hull and A. White paper "Libor vs. OIS: Derivative Discounting Dilemma", Journal of Investment Management, Vol. 11, No. 3, 14-27.

[27] For the US$-denominated risk-free rates, the respective Bloomberg curves are (i) C802 for the US$ Treasury yields; (ii) USSW for US$ Libor swap rates; and (iii) OSSO for OIS rates.

6.1.2 Volatility parameter

Estimation of price volatility depends on the assumptions about the underlying asset value process.

6.1.2.1 Standard Black-Scholes model

Volatility in Black-Scholes formula is estimated based on equation (2.2) as follows

$$\hat{\sigma} = stdev\left[\frac{\Delta \ln S_t}{\sqrt{dt}}\right] \quad (6.1)$$

The equation assumes that the time series S_t of the underlying asset price is available. The equation is a proxy in the case of the price process with mean-reversion property. More generally, the parameter can be estimated based on the following simple linear regression model:

$$\frac{\Delta \ln S_{t+dt}}{\sqrt{dt}} = (\mu - \rho \ln S_t)\sqrt{dt} + \sigma\varepsilon_t \quad (6.2)$$

where parameters μ, ρ and σ are estimated from the auto-regression model described by equation (6.2).

6.1.2.2 Black-Scholes model of leases

6.1.3 Dividend rate

In practice, dividends are discrete and paid at specific dates. Therefore, the equation for the risk-neutral average of the stock price should be represented equivalently as follows:

$$E^Q[S_{t+dt}] = S_t \times [1 + rdt] - D^F \quad (6.3)$$

where $D^F = d \times S \times dt$ is the discrete dividend paid in period t. The dividend payment reduces the price by the dividend value. The risk-neutral price equation can be represented as

$$S_{t+dt} = S_t \times e^{\left(r - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}\varepsilon_t} - D^F$$

6.1.3.1 Discrete dividends: price decomposition

The model with discrete dividends can be interpreted as follows. The dividend-paying asset is decomposed into a portfolio of zero-dividend asset with price S and a portfolio of forward contracts $F^D_{t,T}$, which correspond to each dividend payment D_t. The value of each contract is calculated respectively as

$$F^D_{t,T} = D_t \times e^{r(T-t)}$$

In the model specification, the risk-neutral price is decomposed into two components: (i) risk-neutral price of zero-dividend asset (which is calculated using Black-Scholes formula with $d = 0$) and (ii) a sum of dividend futures calculated as

(6.4) $$D^* = \sum_t D_t \times e^{r(T-t)}$$

where the sum is taken over all discrete dividend paying dates t.

In the case of call/put option estimation, the Black-Scholes formula is implemented as follows.

1. Adjust the strike price by the dividend cash flow as follows:

   (6.5) $$K^* = K + D^*$$

   where K is the option strike price.
2. Apply Black-Scholes call/put equation with $d = 0$ and adjusted strike price K^*.

6.1.3.2 Discrete dividends: conversion of to continuous dividends

Under alternative approach, the cash flow of discrete dividend payments is converted to parameter d which describes continuous dividend payments. The option price is estimated under the approach using standard Black-Scholes equations for continuous dividend payments.

The conversion from discrete to continuous dividends is described by the following equation, assuming dividend payment frequency denoted as m, where $m = 1$ (annual frequency), $m = 2$ (semi-annual frequency), $m = 4$ (quarterly frequency), or $m = 12$ (monthly frequency):

(6.6) $$d = -\frac{1}{T} \times \ln\left[1 - \frac{D_0}{S_0} \times e^{-\frac{r}{m}} \times \frac{1 - e^{-\frac{r \times n}{m}}}{1 - e^{-\frac{r}{m}}}\right]$$

where n is the number of fixed dividend payments over the contract term T. The equation is derived formally in Appendix B.3.

6.2 CDS model

Under the CDS model, the following list of parameters is estimated.

1. Risk-free rate r
2. Dividend rate d
3. Counter-party risk
4. Recovery rate

6.2.1 Dividends rate

In practice, the lease fee (interpreted as dividends) is fixed over the term of the lease contract. Therefore, parameter d is an implied parameter that must be estimated. Similar to the Black-Scholes model, the implied parameter d is estimated from the condition that the risk-neutral price of actual dividends equals to the risk-neutral price of continuous dividends.

In practice, the leases are priced only as of the lease issue date and the lease payments are set as fixed periodic payments over the lease term. To derive the risk-neutral probabilities for the model, we assume that the underlying theoretical model of the lease residual value is the "continuous pricing" model with unknown parameter d, which is estimated from the condition which equates the FMV of the lease payments under the "continuous pricing" and "issue date pricing" models.

Under the "continuous pricing" model, the value of zero-coupon bond is equal to

$$V_t^{zc} = e^{-i \times t}$$

and implied hazard rate is estimated as

$$\gamma = \frac{i - r}{1 - \alpha}$$

Suppose that the market depreciation of the lease asset is described by the following equation

$$F(S) = -\lambda \times S$$

or ($S_t = e^{-\lambda t} \times S_0$). The FMV of continuous and discrete lease payments are calculated respectively as

$$(i + \lambda)S_0 \times \int_0^T e^{-\lambda t} \times e^{-i \times t} \times dt = S_0 \times \left[1 - e^{-(i+\lambda)T}\right] = D_0 \times \sum_i e^{-i \times t_i}$$

where D_0 is a constant periodic lease payment. The unknown parameter i is estimated from the above equation. A necessary condition for the existence of positive solution i is

$$D_0 \times \sum_{t_i} 1 \geq S_0 \times \left[1 - e^{-\lambda T}\right]$$

(Sum of cost-adjusted lease payments over the lease term at least compensates the depreciation of the lease asset over the lease term). In the case of monthly payments, the equation can be represented as follows:

$$S_0 \times \left[1 - e^{-(i+\lambda)T}\right] = D_0 \times \frac{1 - e^{-i \times \frac{n+1}{12}}}{1 - e^{-i \times \frac{1}{12}}}$$

where n is the number of monthly dividend payments over the term of the lease contract.

Section 7 Summary

This section presents the summary of the key results and equations which are spread out across this guide. The section also provides a link to the sections were the equations are derived and discussed in more detail.

Appendix A Diffusion Models

…

A.1 Model specification

A.2 Ito’s lemma

Appendix B Technical Comments

In this section we provide some technical integration formulas used in the models described in this guide.

B.1 Risk-neutral probabilities

Technical derivation of distributions under risk-neutral probabilities are presented below for Black-Scholes models.

Expected value

Risk-neutral probabilities are derived as

(B.1)
$$\begin{cases} q^u & = A^u(1+R) = & \dfrac{S(1+R-D)-S^d}{(S^u-S^d)} \\ q^d & = A^d(1+R) = & \dfrac{-S(1+R-D)+S^u}{(S^u-S^d)} \end{cases}$$

The expected value of the process under risk-neutral probabilities equals

(B.2)
$$E\tilde{S} = q^u \times S^u + q^d \times S^d = S(1+R-D) = S \times [1+(r-d)\times dt]$$

Black-Scholes model

The continuous model is approximated y the following binomial model:

Set of states:

(B.3)
$$\begin{cases} S^u = S^{1-\rho dt} \times \left[1+\left(\mu+\dfrac{\sigma^2}{2}\right)dt+\sigma\sqrt{dt}\right] & p^u = \dfrac{1}{2} \\ S^d = S^{1-\rho dt} \times \left[1+\left(\mu+\dfrac{\sigma^2}{2}\right)dt-\sigma\sqrt{dt}\right] & p^d = \dfrac{1}{2} \end{cases}$$

Risk-neutral probabilities:

(B.4)
$$\begin{cases} q^u = \dfrac{1}{2} + \dfrac{r-d-\mu-\dfrac{\sigma^2}{2}+\rho\ln S_t}{2\sigma} \times \sqrt{dt} \\ q^d = \dfrac{1}{2} - \dfrac{r-d-\mu-\dfrac{\sigma^2}{2}+\rho\ln S_t}{2\sigma} \times \sqrt{dt} \end{cases}$$

The expected value of the process under risk-neutral probabilities equals

$$E\tilde{S} = S \times [1 + (r-d) \times dt]$$

The second moment is estimated as

$$\begin{aligned}
E\tilde{S}^2 &= q^u \times (S^u)^2 + q^d \times (S^d)^2 \\
&= S^{2-2\rho dt} \\
&\times \left[\left(\frac{1}{2} + \frac{r-d-\mu-\frac{\sigma^2}{2}+\rho\ln S_t}{2\sigma} \times \sqrt{dt}\right) \times \left(1 + 2\sigma\sqrt{dt} + 2\left(\mu + \frac{\sigma^2}{2}\right)dt + \sigma^2 dt\right)\right. \\
&+ \left.\left(\frac{1}{2} - \frac{r-d-\mu-\frac{\sigma^2}{2}+\rho\ln S_t}{2\sigma} \times \sqrt{dt}\right) \times \left(1 - 2\sigma\sqrt{dt} + 2\left(\mu + \frac{\sigma^2}{2}\right)dt + \sigma^2 dt\right)\right] \\
&= S^{2-2\rho dt} \times \left[1 + 2\left(\mu + \frac{\sigma^2}{2}\right)dt + \sigma^2 dt + 2\left(r - d - \mu - \frac{\sigma^2}{2} + \rho\ln S_t\right)dt\right] \\
&= S^{2-2\rho dt} \times \left[1 + \left(\sigma^2 + 2(r-d+\rho\ln S_t)\right)dt\right] = S^2 \times [1 - 2\rho\ln S_t\, dt] \times \left[1 + \left(\sigma^2 + 2(r-d+\rho\ln S_t)\right)dt\right] \\
&= S^2 \times [1 + \sigma^2 dt + 2(r-d)dt]
\end{aligned}$$

The risk-neutral variance equals

(B.5) $$Var\,\tilde{S} = E\tilde{S}^2 - \left(E\tilde{S}\right)^2 = S^2 \times \left[1 + \left(\sigma^2 + 2(r-d)\right)dt - 1 - 2(r-d)dt\right] = S^2 \times \sigma^2 \times dt$$

B.2 Derivative pricing

Technical derivation of Black-Scholes model equations is presented below.

Black-Scholes formula

The Black-Scholes formula is derived based on the following equations:

1. $\frac{1}{\sqrt{2\pi T}\sigma} \times \int_{-\infty}^{\infty}\left[e^{\gamma u} \times e^{-\frac{(u-\mu T)^2}{2\sigma^2 T}}\right]du = \frac{1}{\sqrt{2\pi T}\sigma} \times \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2 T}\times\left[u^2 - 2u(\mu T+\gamma\sigma^2 T) + (\mu T+\gamma\sigma^2 T)^2 - 2\mu\gamma\sigma^2 T^2 - \gamma^2\sigma^4 T^2\right]} \times du = e^{\mu\gamma T + \frac{\gamma^2\sigma^2 T}{2}}$;
2. $\frac{1}{\sqrt{2\pi T}\sigma} \times \int_{\ln\frac{K}{S}}^{\infty}\left[e^{\gamma u} \times e^{-\frac{(u-\mu T)^2}{2\sigma^2 T}}\right]du = e^{\mu\gamma T + \frac{\gamma^2\sigma^2 T}{2}} \times \left(1 - \Phi(d)\right) = e^{\mu\gamma T + \frac{\gamma^2\sigma^2 T}{2}} \times \Phi(-d)$, where $d = \frac{\ln\frac{K}{S} - \mu T - \gamma\sigma^2 T}{\sigma\sqrt{T}}$.
3. $E\left[e^{\sigma\sqrt{t}\varepsilon_t}\right] = \frac{1}{\sqrt{2\pi t}\sigma} \times \int_{-\infty}^{\infty}\left[e^{u} \times e^{-\frac{u^2}{2\sigma^2 t}}\right]du = e^{\frac{\sigma^2 t}{2}}$
4. $E\left[\int_0^T e^{\left(r-d-\frac{\sigma^2}{2}\right)\times t + \sigma\sqrt{t}\varepsilon_t} \times dt\right] = \int_0^T e^{\left(r-d-\frac{\sigma^2}{2}\right)\times t + \frac{\sigma^2 t}{2}} \times dt = \int_0^T e^{(r-d)t} \times dt = \frac{1-e^{(r-d)\times T}}{d-r}$.

B.3 Parameter estimation

Technical derivation of the formulas applied for parameter estimation is presented below.

Implied dividend rate in Black-Scholes model

In this section, we derive formally equation (6.6).

(B.6)
$$d = -\frac{1}{T} \times \ln\left[1 - \frac{D_0}{S_0} \times e^{-\frac{r}{m}} \times \frac{1 - e^{-\frac{r \times n}{m}}}{1 - e^{-\frac{r}{m}}}\right]$$

The NPV of the discrete dividend payments equals

$$\sum_{i=1}^{n} D_0 \times e^{-r\frac{i}{m}} = D_0 \times e^{-\frac{r}{m}} \times \frac{1 - e^{-\frac{r \times n}{m}}}{1 - e^{-\frac{r}{m}}}$$

The FMV of the continuous dividend payments estimated under risk-neutral probabilities equals

$$\int_0^T d \times S_0 \times e^{(r-d)t} \times e^{-rt} dt = S_0 \times (1 - e^{-dT})$$

The implied dividend rate d is estimated from the condition that the FMV of continuous dividend payments equals the NPV of discrete dividend payments. The condition is described by equation (B.6).

Implied dividend in lease model

Alternatively, the FMV of the continuous dividend payments estimated based on expected average price depreciation

$$\int_0^T d \times S_0 \times e^{-\lambda t} \times e^{-rt} dt = S_0 \times \frac{d}{r + \lambda}\left(1 - e^{-(r+\lambda)T}\right)$$

The implied dividend rate d is equal to

(B.7)
$$d = \left(\frac{r + \lambda}{1 - e^{-(r+\lambda)T}}\right) \times \left(\frac{D_0}{S_0}\right) \times \left(e^{-\frac{r}{m}} \times \frac{1 - e^{-\frac{r \times n}{m}}}{1 - e^{-\frac{r}{m}}}\right)$$

B.4 ac.APM tool implementation

The ac.APM tool is very similar and is based on the same principals as the ac.SRM tool (which is described in detail in the 'Interest Rate Options' guide). There are however two important conceptual differences.

- ► The ac.APM tool performs the valuation of the risk-neutral probabilities to calculate the value of the modelled instruments. Existence of risk-neutral probabilities requires the markets to be complete. Therefore, the process state must be approximated by a binomial tree;
- ► The ac.APM tool performs both the valuation of the underlying asset (based on the asset generated earnings) and valuation of the derivative instruments. Therefore, derivative modelling includes two components:
- ► Modelling earnings-generating process and estimation of the respective value of the asset; and

► Estimation of the risk-neutral probabilities (based on the asset value model) and estimation of respective derivative prices.

The tool is applied to lease and stock price valuation but can be extended to other examples.

B.4.1 Modelling binomial trees

As discussed above, the ac.APM tool uses a binomial tree to model the underlying process. Note that if the tree has $\frac{T}{dt}$ periods, then the number of states in general growth exponentially, $n = 2^{\frac{T}{dt}}$. In the ac.SRM tool the problem is resolved by using a uniform discrete grid of states and matching the actual states (calculated based on the underlying diffusion process functions) to the nearest discrete states. The transition probabilities must be adjusted to ensure that the mean and standard deviation parameters of the discrete approximation are equal to the actual mean and standard deviation parameters.

The matching of the mean and standard deviation parameters is feasible in general whenever a tree with three or larger number of branches is used. The binomial tree does not allow applying directly the discrete approximation and parameters adjustment.

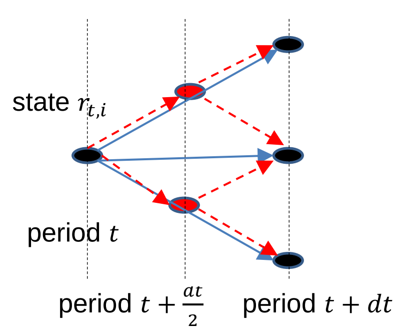
The following workaround to the problem is applied in the ac.APM tool. The binomial tree is constructed in two steps:

1. First, a trinomial tree with step dt is constructed and matched to the discrete states. The trinomial tree is constructed using the methods implemented for the ac.SRM tool.[28] By default, the Hull-White trinomial tree is constructed. The tree is described as follows:

$$dW_t \Rightarrow \begin{cases} \sigma\sqrt{3dt} & p_u \\ 0 & p_m \\ -\sigma\sqrt{3dt} & p_d \end{cases}$$

   where $p_u = p_d = \frac{1}{6}$ and $p_m = \frac{2}{3}$. The trinomial tree models transition distribution with mean zero and standard deviation equal to $\sigma\sqrt{dt}$. The states of the trinomial tree are matched to the nearest discrete states and the probability parameters of the trinomial tree are adjusted to match the mean and standard deviation of the transition distribution of the diffusion process. Note that for small enough dt parameter the adjustment is small and $p_m \geq \frac{1}{2}$ after the adjustment.

2. Second, the trinomial tree is converted to the binomial tree with step $\frac{dt}{2}$ as follows. The steps of the tree construction process are illustrated in the diagram below.



[28] Technical details of the trinomial tree construction are provided in the 'Interest Rate Options' guide.

The black lines show the trinomial tree constructed at step one. The red lines show the binomial extension of the trinomial tree constructed at step two. Existence of the binomial tree (and the required sufficient conditions) is discussed in the next section. Next section also derives the transition probabilities of the binomial tree which match the transition probabilities of the trinomial tree.

B.4.2 Extension of a trinomial into a binomial tree

Lemma: Suppose that $a, b,$ and $1-a-b$ are the probabilities of a trinomial tree ($a > 0, b > 0$, and $1-a-b > 0$). Suppose also that $b \geq \frac{1}{2}$). Then the one-period trinomial tree can be replicated as a two-period binomial tree.

Proof. Suppose that x and y are probabilities of state S^u in the two periods of the binomial tree and that $S^{ud} = S^{du}$ corresponds to the middle state of the trinomial tree. We assume that probability a corresponds to state S^{uu} and probability b corresponds to state $S^{ud} = S^{du}$. The probabilities x and y are estimated from the following system of equations:

$$\begin{cases} xy = a \\ x(1-y) + (1-x)y = b \end{cases}$$

The second equation can be equivalently represented as $x = 2a + b - y$. Substituting the expression for x into the first equation, we obtain the following quadratic equation for y:

$$y^2 - (2a+b)y + a = 0$$

where the solution of the equation must satisfy the following constraints:

$$a \leq y \leq 1$$

Solution of the quadratic equation is described by the following formula:

$$y_{1,2} = \frac{(2a+b) \pm \sqrt{(2a+b)^2 - 4a}}{2}$$

The solutions exist whenever the discriminant of the equation $D = (2a+b)^2 - 4a \geq 0$. If $b \geq \frac{1}{2}$, then $D \geq \left(2a + \frac{1}{2}\right)^2 - 4a = \left(2a - \frac{1}{2}\right)^2 \geq 0$.

Of the two solutions, we pick the larger solution[29]

$$y = \frac{(2a+b) + \sqrt{(2a+b)^2 - 4a}}{2}$$

The constraint $a \leq y$ follows directly from the above equation. Next, we show that the constraint $y \leq 1$ is also satisfied. We show that even a stronger constraint is satisfied: $\frac{(2a+b)+\sqrt{(2a+b)^2-4a}}{2} \leq a + b$. The

[29] Both solutions satisfy the required constraint. The constraint $\frac{(2a+b)-\sqrt{(2a+b)^2-4a}}{2} \geq a$ is equivalent to $b^2 \geq (2a+b)^2 - 4a$, which is equivalent to $1 \geq a + b$.

constraint is equivalent to the following constraint: $(2a+b)^2 \leq b^2 + 4a$ or, equivalently, $4a^2 + 4ab \leq 4a$. The constraint follows directly from the fact that $a + b \leq 1$.

QED

Note that the assumption $b \geq \frac{1}{2}$ was used only to show that $D \geq 0$. This is a sufficient but not a necessary assumption.

Appendix C References

List of literature references used in the notes is provided below

C.1 Lease valuation and hedging

[1] Literature references on lease valuation and hedging models.

[2] David C. Rode, Paul S. Fishbeck, Steve R. Dean, "Residual Risk and the Valuation of Leases under Uncertainty and Limited Information", Carnegie Mellon Electricity Industry Center, CEIC Working Paper 02-02, 2002[30]

[3] Katharina Nau, "An Empirical Analysis of Residual Value Risk in Automotive Lease Contracts", Dissertation, Hohenheim University, Institute of Financial Management, 2012[31]

[4] Sylvain Prado, "Hedging Residual Value Risk Using Derivatives", working paper, UPX, 2009-31, December 2008[32]

C.2 Derivative pricing

[5] CFA, Level I, Volume 6, "Derivative and Alternative Investments", 2013

[6] DerivativeEngines.com, Real Time Option Calculator, http://www.derivativeengines.somee.com/index-3.asp

[7] Financial Risk Manager (FRM), Part I, "Financial Markets and Products", 2016

[8] Financial Risk Manager (FRM), Part I, Book 4, "Valuation and Risk Models", 2014

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[30] The paper is also published as Rode, D., P. Fischbeck, and S. Dean. "Residual Risk and the Valuation of Leases under Uncertainty and Limited Information". Journal of Structured and Project Finance 7:4 (2002): 37-49. Online link to the paper: https://pdfs.semanticscholar.org/c517/301420d394a05698a876adfd74e541504758.pdf

[31] https://core.ac.uk/download/pdf/56707500.pdf

[32] Online link: https://www.researchgate.net/publication/254406042_Hedging_residual_value_risk_using_derivatives